

# Extinction probability and total progeny of predator-prey dynamics on infinite trees

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## Abstract

We consider the spreading dynamics of two nested invasion clusters on an infinite tree. This model was defined as the chase-escape model by Kordzakhia and it admits a limit process, the birth-and-assassination process, previously introduced by Aldous and Krebs. On both models, we prove an asymptotic equivalent of the extinction probability near criticality. In the subcritical regime, we give a tail bound on the total progeny of the preys before extinction.

**Keywords:** SIR models, predator-prey dynamics, branching processes.

## 1 Introduction

The *chase-escape process* is a stochastic predator-prey dynamics which was studied by Kordzakhia [14] on a regular tree. In a previous paper, Aldous and Krebs [4] had introduced the *birth-and-assassination (BA) process*. The latter model can be seen as a natural limit of the chase-escape model. In [8] the two models were merged into the *rumor scotching process*. The original motivation of Aldous and Krebs was then to analyze a scaling limit of a queueing process with blocking which appeared in database processing, see Tsitsiklis, Papadimitriou and Humblet [22]. As pointed in [8], the BA process is also the scaling limit of a rumor spreading model which is motivated by network epidemics and dynamic data dissemination (see for example, [17], [5], [18]).

We may conveniently define the chase-escape processes as a SIR dynamics (see for example [17] or [5] for some background on standard SIR dynamics). This process represents the dynamics of a rumor/epidemic spreading on the vertices of a graph along its edges. A vertex may be unaware of the rumor/susceptible (S), aware of the rumor and spreading it as true/infected (I), or aware of the rumor and trying to scotch it/recovered (R).

We fix a locally finite connected graph  $G = (V, E)$ . The chase-escape process is described by a Markov process on  $\mathcal{X} = \{S, I, R\}^V$ . If  $\{u, v\} \in E$ , we write  $u \sim v$ . For  $v \in V$ , we also define the  $\mathcal{X} \rightarrow \mathcal{X}$  maps  $I_v$  and  $R_v$  by : for  $x = (x_u)_{u \in V}$ ,  $(I_v(x))_u = (R_v(x))_u = x_u$ , if  $u \neq v$  and  $(I_v(x))_v = I$ ,  $(R_v(x))_v = R$ . Let  $\lambda \in (0, 1)$  be a fixed infection intensity. We then define the Markov process with transition rates:

$$\begin{aligned} K(x, I_v(x)) &= \lambda \mathbf{1}(x_v = S) \sum_{u \sim v} \mathbf{1}(x_u = I), \\ K(x, R_v(x)) &= \mathbf{1}(x_v = I) \sum_{u \sim v} \mathbf{1}(x_u = R), \end{aligned}$$

and all other transitions have rate 0. The absorbing states of this process are the states without  $I$ -vertices or with only  $I$  vertices. In this paper, we are interested by the behavior of the process when at time 0 there is a non-empty finite set of  $I$  and  $R$ -vertices.

In [14], this model was described as a predator-prey dynamics: each vertex may be empty (S), occupied by a prey (I) or occupied by a predator (R). The preys spread on unoccupied vertices and predators spread on vertices occupied by preys. If  $G$  is the  $\mathbb{Z}^d$ -lattice and if there is no  $R$ -vertices, the process is the original Richardson's model [19]. With  $R$ -vertices, this process is a variant of the two-species Richardson model with prey and predators, see for example Häggström and Pemantle [12], Kordzakhia and Lalley [15].

The chase-escape process differs from the classical SIR dynamics on the transition from  $I$  to  $R$ : in the classical SIR dynamics, a  $I$ -vertex is recovered at rate 1 independently of its neighborhood.

**Chase-escape process on a tree** If the graph  $G = T = (V, E)$  is a rooted tree, the process is much simpler to study. We denote by  $\emptyset$  the root of  $T$ . For the range of initial conditions of interest (non-empty finite set of  $I$  and  $R$ -vertices), there is no real loss of generality to study the chase-escape process on the tree  $T^\downarrow$  obtained from  $T$  by adding a particular vertex, say  $o$ , connected to the root of the tree. At time 0, vertex  $o$  is in state  $R$ , the root  $\emptyset$  is in state  $I$ , while all other vertices are in state  $S$  (see figure 1). We shall denote by  $X(t) \in \{S, I, R\}^V$  our Markov process on the tree  $T^\downarrow$ . Under  $\mathbb{P}_\lambda$ ,  $X$  is the chase escape process on  $T^\downarrow$  with infection rate  $\lambda$ .

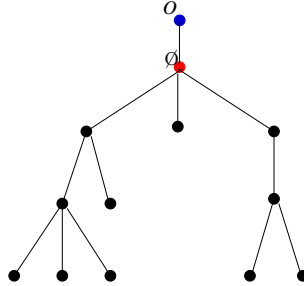


Figure 1: The initial condition : the root is  $I$ ,  $o$  is  $R$ , all other vertices are  $S$ .

We say that the Markov process  $X$  *gets extinct* if at some (random) time  $\tau < \infty$ , there is no  $I$ -particle. Otherwise the process is said to *survive*. We define the probability of extinction as

$$q_T(\lambda) = \mathbb{P}_\lambda(X \text{ gets extinct}).$$

Obviously, if  $T$  is finite then  $q_T(\lambda) = 1$  for any  $\lambda \geq 0$ . Before stating our results, we first need to introduce some extra terminology.

There is a canonical way to represent the vertex set  $V$  as a subset of  $\mathbb{N}^f = \cup_{k=0}^{\infty} \mathbb{N}^k$  with  $\mathbb{N}^0 = \emptyset$  and  $\mathbb{N} = \{1, 2, \dots\}$ . If  $k \geq 1$  and  $v \in V$  is at distance  $k$  from the root, then  $v = (i_1, \dots, i_k) \in V \cap \mathbb{N}^k$ . The *genitor* of  $v$  is  $(i_1, \dots, i_{k-1})$ : it is first vertex on the path from  $v$  to the root  $\emptyset$  of length  $k$ . The *offsprings* of  $v$  are set of vertices who have genitor  $v$ . They are indexed by  $(i_1, \dots, i_k, 1), \dots, (i_1, \dots, i_k, n_v)$ , where  $n_v$  is the number of offsprings of  $v$ . The *ancestors* of  $v$  is the set of vertices  $(i_1, \dots, i_\ell)$ ,  $0 \leq \ell \leq k-1$  with the convention

$i_0 = \emptyset$ . Similarly, the  $n$ -th generation offsprings of  $v$  are the vertices in  $V \cap \mathbb{N}^{k+n}$  of the form  $(v, i_{k+1}, \dots, i_{k+n})$ .

Recall that the *upper growth rate*  $d \in [1, \infty]$  of a rooted infinite tree  $T$  is defined as

$$d = \limsup_{k \rightarrow \infty} |V_k|^{1/k},$$

where  $V_k = V \cap \mathbb{N}^k$  is the set of vertices at distance  $k$  from the root  $\emptyset$  and  $|\cdot|$  denotes the cardinal of a finite set. The *lower growth rate* is defined similarly with a  $\liminf$ . When the  $\liminf$  and the  $\limsup$  coincide, this defines the *growth rate* of the tree.

For example, for integer  $d \geq 1$ , we define the  $d$ -ary tree as the tree where all vertices have exactly  $d$  offsprings<sup>1</sup>. Obviously, the  $d$ -ary tree has growth rate  $d$ . More generally, consider a realization  $T$  of a Galton-Watson tree with mean number of offsprings  $d \in (1, \infty)$ . Then, the Seneta-Heyde Theorem [21, 13] implies that, conditioned on  $T$  infinite, the growth rate of  $T$  is a.s. equal to  $d$ . For background on random trees and branching processes, we refer to [6, 20].

For integer  $n \geq 1$ , we define  $T^{*n}$  as the rooted tree on  $V$  obtained from  $T$  by putting an edge between all vertices and their  $n$ -th generation offsprings. For real  $d > 1$ , we say that  $T$  is a *lower  $d$ -ary* if for any  $1 < \delta < d$ , there exist an integer  $n \geq 1$  and  $v \in V$  such that the subtree of the descendants of  $v$  in  $T^{*n}$  contains a  $\lceil \delta^n \rceil$ -ary tree. Note that if  $T$  is lower  $d$ -ary then its lower growth rate is at least  $d$ . Also, if  $T$  is the realization of a Galton-Watson tree with mean number of offsprings  $d \in (1, \infty)$  then, conditioned on  $T$  infinite,  $T$  is a.s. lower  $d$ -ary (for a proof see Lemma 23 in appendix).

The first result is an extension of [14, Theorem 1] where it is proved for  $d$ -ary trees. It describes the phase transition of the event of survival.

**Theorem 1.** *Let  $d > 1$  and*

$$\lambda_1 = 2d - 1 - 2\sqrt{d(d-1)}.$$

*If  $0 < \lambda < \lambda_1$  and the upper growth rate of  $T$  is at most  $d$ , then  $q_T(\lambda) = 1$ . If  $\lambda > \lambda_1$  and  $T$  is lower  $d$ -ary, then  $0 < q_T(\lambda) < 1$ .*

Note that in the classical SIR dynamics, it is easy to check that the critical value of  $\lambda$  is  $\lambda = 1/(d-1)$ . Also, for any  $d > 1$ ,  $\lambda_1 < 1$  and,

$$\lambda_1 \sim_{d \uparrow \infty} \frac{1}{4d}. \tag{1}$$

The proof of Theorem 1 will follow a strategy parallel to [14, 4]. We employ techniques akin to the study the infection process in the Richardson model. They will be based on large deviation estimates on the probability that a single vertex is  $I$  at time  $t$ .

To our knowledge, there is no known closed form expression for the extinction probability  $q_T(\lambda)$ . Our next result determines an asymptotic equivalent for the probability of survival for  $\lambda$  close to  $\lambda_1$ . Our method does not seem to work on the sole assumption that  $T$  has growth rate  $d > 1$  and is lower  $d$ -ary. We shall assume that  $T$  is a realization of a Galton-Watson tree with offspring distribution  $\pi$  and

$$d = \sum_{k=1}^{\infty} k\pi(k) > 1.$$

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<sup>1</sup>It would be more proper to call this tree the complete infinite  $d$ -ary tree.

We consider the annealed probability of extinction:

$$q(\lambda) = \mathbb{E}'[q_T(\lambda)] = \mathbb{P}'_\lambda(X \text{ gets extinct}),$$

where the expectation  $\mathbb{E}'(\cdot)$  is with respect to the randomness of the tree and  $\mathbb{P}'_\lambda(\cdot) = \mathbb{E}'(\mathbb{P}_\lambda(\cdot))$  is the probability measure with respect to the joint randomness of  $T$  and  $X$ . Note that in the specific case  $d$  integer and  $\pi(d) = 1$ ,  $T$  is the  $d$ -ary tree and the measures  $\mathbb{P}'_\lambda$  and  $\mathbb{P}_\lambda$  coincide.

**Theorem 2.** *Assume further that the offspring distribution has finite second moment. There exist constants  $c_0, c_1 > 0$  such that for all  $\lambda_1 < \lambda < 1$ ,*

$$c_0 \omega^3 e^{-\frac{(1-\lambda_1)\pi}{2(d(d-1))^{1/4}} \omega^{-1}} \leq 1 - q(\lambda) \leq c_1 e^{-\frac{(1-\lambda_1)\pi}{2(d(d-1))^{1/4}} \omega^{-1}},$$

with

$$\omega = \sqrt{\lambda - \lambda_1}.$$

Note that the behavior depicted in Theorem 2 contrasts with the classical SIR dynamics, where  $1 - q(\lambda)$  is of order  $(\lambda(d-1) - 1)_+$ . This result should however be compared to similar results in the Brunet-Derrida model of branching random walk killed below a linear barrier, see Gantert, Hu and Shi [11] and also Bérard and Gouéré [7]. As in this last reference, our approach is purely analytic. We will first check that  $q(\lambda)$  is related to a second order non-linear differential equation. Then, we will rely on comparisons with linear differential equations. A similar technique was already used by Brunet and Derrida [9], and notably also in Mueller, Mytnik and Quastel [16, section 2]. The parallel with the Brunet-Derrida model of branching random walk killed below a linear barrier is that we may think the growing recovery cluster as a randomly growing barrier.

In the case  $0 < \lambda < \lambda_1$ , the process  $X$  stops a.s. evolving after some finite  $\tau$ . We define  $Z$  as the total number of recovered vertices (excluding the vertex  $o$  of  $T^\downarrow$ ) at time  $\tau$ . It is the number of vertices which have been infected before the process reaches its absorbing state. We define the annealed parameter:

$$\gamma(\lambda) = \sup \{u \geq 0 : \mathbb{E}'_\lambda[Z^u] < \infty\}.$$

The scalar  $\gamma(\lambda)$  can be thought as a power-tail exponent of the variable  $Z$  under the annealed measure  $\mathbb{P}'_\lambda$ . In particular, for any  $0 < \gamma < \gamma(\lambda)$ , from Markov Inequality, there exists a constant  $c > 0$  such that for all  $t \geq 1$ ,  $\mathbb{P}'_\lambda(Z > t) \leq ct^{-\gamma}$ . We define

$$\gamma_\pi = \sup \left\{ u \geq 1 : \sum_{k=1}^{\infty} k^u \pi(k) < \infty \right\} \geq 1.$$

**Theorem 3.** (i) *For any  $0 < \lambda < \lambda_1$ ,*

$$\gamma(\lambda) = \min \left( \frac{\lambda^2 - 2d\lambda + 1 - (1-\lambda)\sqrt{\lambda^2 - 2\lambda(2d-1) + 1}}{2\lambda(d-1)}, \gamma_\pi \right).$$

(ii) *Let  $1 \leq u < \gamma_\pi$ ,  $A_u = u^2(d-1) + 2ud + (d-1)$ , and*

$$\lambda_u = \frac{A_u - \sqrt{A_u^2 - 4u^2}}{2u}.$$

*If  $\lambda < \lambda_u$  then  $\mathbb{E}'_\lambda[Z^u]$  is finite. If  $\lambda > \lambda_u$ ,  $\mathbb{E}'_\lambda[Z^u]$  is infinite.*

It is straightforward to check that (i) is equivalent to (ii). This result contrasts with classical SIR dynamics. For example, if  $T$  is the  $d$ -ary tree, for all  $\lambda < 1/(d-1)$  there exists a constant  $c > 0$  such that  $\mathbb{E}'_\lambda \exp(cZ) < \infty$ . Here, the heavy-tail phenomenon is an interesting feature of the chase-escape process. It also appears in the Brunet-Derrida model, see Addario-Berry and Broutin [1], Aïdékon [2] and Aïdékon Hu and Zindy [3]. Note finally that

$$\gamma(\lambda) \sim_{\lambda \downarrow 0} \min\left(\frac{1}{(d-1)\lambda}, \gamma_\pi\right) \quad \text{and} \quad \gamma(\lambda) \sim_{\lambda \uparrow \lambda_1} 1.$$

By recursion, we will also compute the moments of  $Z$ . The computation of the first moment gives

**Theorem 4.** *If  $0 < \lambda \leq \lambda_1$  and  $\Delta = \lambda^2 - 2\lambda(2d-1) + 1$ , then*

$$\mathbb{E}'_\lambda[Z] = \frac{2d}{(d-1)(1+\lambda+\sqrt{\Delta})} - \frac{1}{d-1}.$$

Theorem 4 implies a surprising right discontinuity of the function  $\lambda \mapsto \mathbb{E}'_\lambda Z$  at the critical intensity  $\lambda = \lambda_1$ :  $\mathbb{E}'_{\lambda_1} Z = 2d/((d-1)(1+\lambda_1)) - 1/(d-1) < \infty$ . Again, this discontinuity contrasts with what happens in a standard Galton-Watson process near criticality, where for  $0 < \lambda < 1/(d-1)$ ,  $\mathbb{E}'_\lambda Z$  is of order  $(1 - (d-1)\lambda)^{-1}$ . From Theorem 4, we may fill the gap in Theorem 1 in the specific case of a realization of a Galton-Watson tree.

**Corollary 5.** *Let  $T$  be a Galton-Watson tree with mean number of offsprings  $d$ . Then a.s.  $q_T(\lambda_1) = 1$ .*

The method of proofs of Theorems 3-4 will be parallel to arguments in [8] on the birth-and-assassination process.

**The birth-and-assassination process** We now turn to the BA process. It is a scaling limit in  $d \rightarrow \infty$  of the chase-escape process on the  $d$ -ary tree when  $\lambda$  is rescaled in  $\lambda/d$ .

Informally, the process can be described as follows. We start from a root vertex that produces offsprings according to a Poisson process of rate  $\lambda$ . Each offspring in turn produces children according to independent Poisson processes and so on. The children of the root are said to belong to the first generation and their children to the second generation and so forth. Independently, the root vertex is *at risk* at time 0 and dies after a random time  $D_\emptyset$  that is exponentially distributed with mean one. Its offsprings become at risk after time  $D_\emptyset$  and the process continues in the next generations. In other words, conditioned on  $D_\emptyset$ , each offspring has a independent exponentially distributed life time with mean one. We now make precise the above description.

As above,  $\mathbb{N}^f = \cup_{k=0}^\infty \mathbb{N}^k$  denotes the set of finite  $k$ -tuples of positive integers (with  $\mathbb{N}^0 = \emptyset$ ). Elements from this set are used to index the offspring in the BA process. Let  $\{\Xi_v\}, v \in \mathbb{N}^f$ , be a family of independent Poisson processes with common arrival rate  $\lambda$ ; these will be used to define the offsprings. Let  $\{D_v\}, v \in \mathbb{N}^f$ , be a family of independent, identically distributed (iid) exponential random variables with mean 1; we use them to assign the lifetime for the appropriate offspring. The families  $\{\Xi_v\}$  and  $\{D_v\}$  are independent. The process starts at time 0 with only the root, indexed by  $\emptyset$ . This produces offspring at the arrival times determined by  $\Xi_\emptyset$  that enter the system with indices (1), (2),  $\dots$  according to

their birth order. Each new vertex  $v$ , immediately begins producing offspring determined by the arrival times of  $\Xi_v$ . The offspring of  $v$  are indexed  $(v, 1), (v, 2), \dots$  also according to birth order. The root is *at risk* at time 0. It continues to produce offspring until time  $T_\emptyset = D_\emptyset$ , when it dies. Let  $k > 0$  and let  $v = (n_1, \dots, n_{k-1}, n_k)$ ,  $v' = (n_1, \dots, n_{k-1})$  denote a vertex and its genitor. When a particle  $v'$  dies (at time  $T_{v'}$ ), the particle  $v$  then becomes at risk; it in turn continues to produce offspring until time  $T_v = T_{v'} + D_v$ , when it dies (see figure 2).

The BA process can be equivalently described as a Markov process  $X(t)$  on  $\{S, I, R\}^{\mathbb{N}^f}$ , where a particle/vertex in state  $S$  is not yet born, a particle in state  $I$  is alive and a particle in state  $R$  is dead. A particle is at risk if it is in state  $I$  and its genitor is in state  $R$ . We use the same notation than above : under  $\mathbb{P}_\lambda$ , the process  $X(t)$  has infection rate  $\lambda > 0$ ,  $q(\lambda)$  is the probability of extinction and so on.

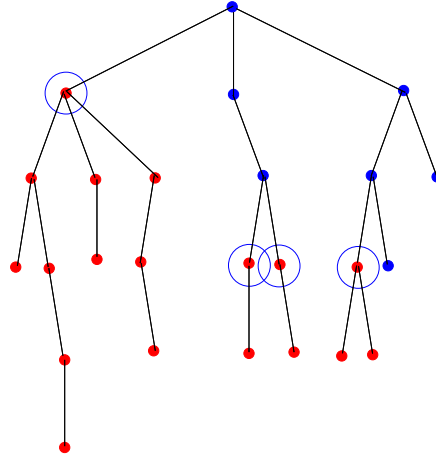


Figure 2: Illustration of the birth-and-assassination process, living particles are in red, dead particles in blue, particles at risk are encircled.

The following result from [4] describes the phase transition on the probability of survival as a function of  $\lambda$ .

**Theorem 6.** *Consider the BA process with rate  $\lambda > 0$ . If  $\lambda \in [0, 1/4]$ , then  $q(\lambda) = 1$ , while if  $\lambda > \frac{1}{4}$ ,  $0 < q(\lambda) < 1$ .*

The critical case  $\lambda = 1/4$  was established in [8]. Note also that the threshold  $\lambda = 1/4$  is consistent with (1).

Our final result is the analog of Theorem 2.

**Theorem 7.** *Consider the BA process and assume that  $\lambda > 1/4$ . There exist constants  $c_0, c_1 > 0$  such that for all  $1/4 < \lambda < 1$ ,*

$$c_0 \omega^3 e^{-\frac{\pi}{2} \omega^{-1}} \leq 1 - q(\lambda) \leq c_1 \omega^{-1} e^{-\frac{\pi}{2} \omega^{-1}},$$

with

$$\omega = \sqrt{\lambda - \frac{1}{4}}.$$

Note that the analog of Theorems 3-4 was already performed in [8]. The remainder of the paper is organized as follows. In section 2, we prove Theorem 1. In section 3, we prove Theorem 2. In section 4, we prove Theorems 3-4. Finally, in section 5, we extend our previous arguments to the BA process and prove Theorem 7. The proof of Theorem 7 can be read independently. It is slightly simpler than the proof of Theorem 2.

## 2 Proof of Theorem 1

We define the set recovered and infected vertices as  $R(t) = \{v \in V : X_v(t) = R\}$  and  $I(t) = \{v \in V : X_v(t) = I\}$ . The set  $R(t)$  being non-decreasing, we may define  $R(\infty) = \cup_{t>0} R(t)$  and  $Z = |R(\infty)| \in \mathbb{N} \cup \{\infty\}$ . Note that also a.s.  $R(\infty) = \{v \in V : \exists t > 0, X_v(t) = I\}$ , in words,  $R(\infty)$  is the set of vertices which have been infected at some time.

### 2.1 Subcritical regime

We fix  $0 < \lambda < \lambda_1$ . In order to prove that  $q_T(\lambda) = 1$ , it is sufficient to prove that  $\mathbb{E}_\lambda Z < \infty$ . Let  $V_k$  be the set of vertices of  $V$  which are at distance  $k$  from the root  $\phi$ . The time for infection of  $v \in V_n$  is the sum of  $n$  i.i.d.  $\text{Exp}(\lambda)$  variables say  $\xi_1, \dots, \xi_n$ . The time for recovery of the ancestor of  $v$  is the sum of  $n$  i.i.d.  $\text{Exp}(1)$  variables, say  $D_1, \dots, D_n$ , independent of  $(\xi_1, \dots, \xi_n)$ . Let  $v_0, \dots, v_n$  be the ancestor line of  $v$ :  $v_0 = \phi$  and  $v_n = v$ . The vertex  $v$  will have been infected if for all  $1 \leq m \leq n$ ,  $v_{m-1}$  has infected  $v_m$  before being recovered. We thus find

$$\mathbb{P}_\lambda(v \in R(\infty)) = \mathbb{P}_\lambda\left(\forall 1 \leq m \leq n, \sum_{i=1}^m \xi_i < \sum_{i=1}^m D_i\right) \leq \mathbb{P}_\lambda\left(\sum_{i=1}^n \xi_i < \sum_{i=1}^n D_i\right).$$

The Chernov bound gives for any  $0 < \theta < \lambda$ ,

$$\begin{aligned} \mathbb{P}_\lambda\left(\sum_{i=1}^n \xi_i < \sum_{i=1}^n D_i\right) &\leq \mathbb{E}_\lambda \exp\left\{\theta\left(\sum_{i=1}^n \xi_i - \sum_{i=1}^n D_i\right)\right\} \\ &= \left(\frac{\lambda}{\lambda - \theta}\right)^n \left(\frac{1}{1 + \theta}\right)^n, \end{aligned}$$

where we have used the independence of all variables at the last line. Now, the above expression is minimized for  $\theta = (\lambda - 1)/2$ . We find

$$\mathbb{P}_\lambda(v \in R(\infty)) \leq \left(\frac{4\lambda}{(\lambda + 1)^2}\right)^n.$$

Also, by assumption,

$$|V_n| \leq (d + o(1))^n.$$

It follows that

$$\mathbb{E}_\lambda Z = \sum_{v \in V} \mathbb{P}_\lambda(v \in R(\infty)) \leq \sum_{n=0}^{\infty} \left(\frac{4(d + o(1))\lambda}{(\lambda + 1)^2}\right)^n.$$

It is now straightforward to check that

$$\frac{4d\lambda}{(\lambda + 1)^2} < 1,$$

if  $\lambda < \lambda_1$ . This concludes the first part of the proof.

## 2.2 Supercritical regime

We now fix  $\lambda > \lambda_1$ . We should prove that  $q_T(\lambda) < 1$ . The monotony of the process in  $\lambda$  implies that we may assume without generality that  $\lambda_1 < \lambda < 1$ . For  $\delta > 0$ , we define the function  $g_\delta$  by, for all  $x > 0$ ,

$$\begin{aligned} g_\delta(x) &= \frac{1}{x} - \log\left(\frac{1}{x}\right) + \frac{\lambda}{x} - \log\left(\frac{\lambda}{x}\right) - 2 - \log(\delta) \\ &= \frac{1+\lambda}{x} + \log\left(\frac{x^2}{\lambda\delta}\right) - 2. \end{aligned}$$

Taking derivative, the minimum of  $g_\delta$  is reached at  $c = (1+\lambda)/2$ . We deduce easily the following property of the function  $g_\delta$ .

**Lemma 8.** *If  $\lambda_1 < \lambda < 1$ ,  $\min_{x>0} g_\delta(x) < 0$ .*

By Lemma 8, using continuity, we deduce that there exist  $c > 0$  and  $1 < \delta < d$  such that

$$g_\delta(c) < 0.$$

In the remainder of the proof, we fix such pair  $(c, \delta)$ .

**Construction of a nested branching process.** We fix an integer  $m \geq 1$  that we will be completely specified later on. We assume that  $m$  is large enough such that  $T^{*m}$  contains a  $\lceil \delta^m \rceil$ -ary subtree. We denote by  $T'$  this subtree and by  $x \in V$  its root. For integer  $k \geq 0$ , we define  $V'_k$  as the set of vertices of generation  $k$  in  $T'$ . Note that by assumption

$$|V'_k| = \lceil \delta^m \rceil^k.$$

We may assume that the generation of  $x$  in  $T$  is larger than  $m$ . We denote  $a(x) \in V$  the  $m$ -th ancestor of  $x$  in  $T$ . For  $z \in V'_k$  and  $k \geq 1$ , we denote by  $a(z) \in V'_{k-1}$  its ancestor in  $T'$ . For example, if  $z \in V'_1$ ,  $a(z) = x$ .

The chase-escape process can be constructed thanks to i.i.d.  $\text{Exp}(\lambda)$  variables  $(\xi_v)_{v \in V}$  and independent i.i.d.  $\text{Exp}(1)$  variables  $(D_v)_{v \in V}$ . The variable  $\xi_v$  (resp.  $D_v$ ) is the time by which  $v \in V$  will be infected (resp. recovered) once its ancestor is infected (resp. recovered).

We now start a branching process as follows. We set  $x$  to be the root of the process,  $\mathcal{S}_0 = \{x\}$ . We define recursively the offsprings of the  $k$ -th generation as the set  $\mathcal{S}_k$  of vertices  $z \in V'_k$  satisfying the following three conditions :

1. the vertex  $a(z) \in V'_{k-1}$  belongs to  $\mathcal{S}_{k-1}$ ;
2.  $\sum_{i \in \pi} \xi_i \leq \frac{m}{c}$  where  $\pi$  is the set of the vertices on the path from  $a(z)$  to  $z$  (excluding  $a(z)$ );
3.  $\sum_{i \in \pi} D_i \geq \frac{m}{c}$  where  $\pi$  is the set of the vertices on the path from  $a(a(z))$  to  $a(z)$  (excluding  $a(z)$ ).

Thus for  $z \in V'_k$ , such that its ancestor  $a(z) \in \mathcal{S}_{k-1}$ , we have that

$$\mathbb{P}_\lambda(z \in \mathcal{S}_k | \mathcal{S}_{k-1}) = \mathbb{P}_\lambda\left(\sum_{i=1}^m \xi_i \leq \frac{m}{c}\right) \mathbb{P}_\lambda\left(\sum_{i=1}^m D_i \geq \frac{m}{c}\right).$$



Notice that by construction, the number of offsprings of  $z \neq z'$  in  $\mathcal{S}_{k-1}$  are identically distributed and they are independent if  $a(z) \neq a(z')$ . It follows that the process forms a multi-type branching process with a finite number of types (one for each possible configuration of  $\mathcal{S}_1$ ). In the next paragraph, we will check that this branching process is supercritical, i.e. we will prove that

$$M = \sum_{v \in V'_1} \mathbb{P}_\lambda(z \in \mathcal{S}_1) > 1.$$

It implies that with positive probability, the branching process does not die out (see Athreya and Ney [6, chapter 5]).

We first notice the following. Assume that at some time  $t > 0$ , the vertex  $x$  becomes infected and that  $a(x)$  is still infected. Note that the existence of such finite time  $t > 0$  has positive probability. Let us denote by  $E$  such event. We set  $t_0 = t$  and, for integer  $k \geq 1$ ,

$$t_k = t_{k-1} + \frac{m}{c}.$$

By construction, if  $E$  holds and  $z \in \mathcal{S}_k$  then, at time  $t_k$ ,  $z$  and  $a(z)$  are both infected. It follows that in order to conclude the proof of Theorem 1 it is sufficient to prove that  $M > 1$ .

**The nested branching process is supercritical.** We need a standard large deviation estimate. We define

$$J(x) = x - \log x - 1.$$

The next lemma is an immediate consequence of Cramer's Theorem for exponential variables (see [10, §2.2.1]).

**Lemma 9.** *Let  $(\xi_i)_{i \geq 1}$ , be i.i.d.  $\text{Exp}(\lambda)$  variables. For any  $a > 1/\lambda$ , we have that*

$$\liminf_{m \rightarrow \infty} \frac{1}{m} \log \mathbb{P} \left( \sum_{i=1}^m \xi_i \geq am \right) \geq -J(\lambda a),$$

while, for any  $a < 1/\lambda$ ,

$$\liminf_{m \rightarrow \infty} \frac{1}{m} \log \mathbb{P} \left( \sum_{i=1}^m \xi_i \leq am \right) \geq -J(\lambda a).$$

Note that the bounds of Lemma 9 hold for all  $a > 0$  (even if they are sharp only for the above ranges). We may now estimate the terms in (2). We have from Lemma 9 that

$$\mathbb{P}_\lambda \left( \sum_{i=1}^m \xi_i \leq \frac{m}{c} \right) \geq \exp \left\{ -mJ \left( \frac{\lambda}{c} \right) + o(m) \right\}$$

and

$$\mathbb{P}_\lambda \left( \sum_{i=1}^m D_i \geq \frac{m}{c} \right) \geq \exp \left\{ -mJ \left( \frac{1}{c} \right) + o(m) \right\}.$$

Thus we obtain a lower bound on the mean number of offspring in the first generation to be

$$\begin{aligned}
M &= \sum_{z \in V'_1} \mathbb{P}_\lambda(z \in \mathcal{S}_1) \\
&\geq \lceil \delta^m \rceil \exp \left\{ -m \left( J \left( \frac{1}{c} \right) + J \left( \frac{\lambda}{c} \right) + o(m) \right) \right\} \\
&\geq \exp(-mg_\delta(c) + o(m)),
\end{aligned}$$

where  $g_\delta(\cdot)$  is as defined in (2). If  $m$  was chosen large enough, we have that  $M > 1$  and hence that the branching process is supercritical. Therefore with positive probability, this branching process does not die out. This proves the theorem.  $\blacksquare$

### 3 Proof of Theorem 2

#### 3.1 Differential equation for the survival probability

We first determine a differential equation associated to the probability of extinction. Under  $\mathbb{P}'_\lambda$ , define  $Q_\lambda(t)$  to be the extinction probability given that the root  $\emptyset$  is recovered at time  $t \geq 0$  so that

$$q(\lambda) = \int_0^\infty Q_\lambda(t) e^{-t} dt \quad (2)$$

and  $Q_\lambda(0) = 1$ .

Now, in  $T$ , the offsprings of the root are  $\{1, \dots, N\}$ , where  $N$  has distribution  $\pi$ . The root infects each of its offspring after an independent exponential variable with intensity  $\lambda$ . Let  $\{\xi_i\}_{1 \leq i \leq N}$  be the infection times. Note that in  $T$ , the subtrees generated by each of the offsprings of the root are iid copies of  $T$ . Hence, if for integer  $i$  with  $1 \leq \xi_i \leq D_\emptyset$ , we define  $X^i$  as the subprocess on vertices  $(i\mathbb{N}^f) \cap V$  with ancestors  $i$ . Conditioned on  $D_\emptyset = t$ , on  $N$  and  $(\xi_i)_{1 \leq i \leq N}$ , the processes  $(X^i)$  are independent chase-escape processes conditioned on the root becomes at risk at time  $t - \xi_i$  (where we say that a  $I$ -vertex is at risk if its genitor is in state  $R$ ).

For the process  $X$  to get extinct, all the processes  $X^i$  must get extinct. So finally, we get

$$Q_\lambda(t) = \mathbb{E}'_\lambda \left[ \prod_{1 \leq i \leq N} (\mathbf{1}(\xi_i > t) + \mathbf{1}(\xi_i \leq t) Q_\lambda(t - \xi_i + D_i)) \right]$$

where  $(D_i), i \geq 1$ , are independent exponential variables with parameter 1. Consider the generating function of  $\pi$

$$\psi(x) = \mathbb{E}'_\lambda[x^N] = \sum_{k=0}^\infty x^k \pi(k).$$

Recall that  $\psi$  is strictly increasing and convex on  $[0, 1]$  and  $\psi'(1) = \mathbb{E}N = d$ . We find, for any

$t \geq 0$ ,

$$\begin{aligned}
Q_\lambda(t) &= \psi \left( e^{-\lambda t} + \lambda \int_0^t e^{-\lambda x} \int_0^\infty Q_\lambda(t-x+s) e^{-s} ds dx \right) \\
&= \psi \left( e^{-\lambda t} + \lambda e^{-\lambda t} \int_0^t e^{\lambda x} \int_0^\infty Q_\lambda(x+s) e^{-s} ds dx \right) \\
&= \psi \left( e^{-\lambda t} + \lambda e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty Q_\lambda(s) e^{-s} ds dx \right).
\end{aligned}$$

Performing the change of variable

$$x(t) = \psi^{-1}(Q_\lambda(t)) \in [0, 1], \quad (3)$$

yields to

$$x(t) = e^{-\lambda t} + \lambda e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty \psi(x(s)) e^{-s} ds dx. \quad (4)$$

We multiply the above expression by  $e^{\lambda t}$  and differentiate once, it gives

$$e^{\lambda t}(\lambda x(t) + x'(t)) = \lambda e^{(\lambda+1)t} \int_t^\infty \psi(x(s)) e^{-s} ds, \quad (5)$$

Now, multiplying the above expression by  $e^{-(\lambda+1)t}$  and differentiating once again, we find that  $x(t)$  satisfies the differential equation

$$x'' - (1 - \lambda)x' + \varphi(x) = 0 \quad (6)$$

with

$$\varphi(x) = \lambda \psi(x) - \lambda x.$$

This non-linear ordinary differential equation is not easy to solve. However, in the neighborhood of  $\lambda = \lambda_1$  it is possible to obtain an asymptotic expansion as explained below.

### 3.2 A fixed point equation

We define  $\rho \in [0, 1)$  as the extinction probability in the Galton-Watson tree:

$$\rho = \psi(\rho).$$

We note that  $\varphi$  is convex,  $\varphi(1) = \varphi(\rho) = 0$ ,  $\varphi$  is negative on  $(\rho, 1)$  and it is increasing in a neighborhood of 1,  $\varphi'(1) = \lambda(d-1) > 0$ .

Let  $\mathcal{H}$  be the set of non-increasing functions  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $f(0) = 1$ ,  $\lim_{t \rightarrow \infty} f(t) = \rho$ . The next lemma is an easy consequence of the monotony of the process.

**Lemma 10.** *For any  $\lambda > \lambda_1$ , the function  $x(\cdot)$  defined by (3) is in  $\mathcal{H}$ .*

*Proof:* The monotony of the process implies that  $x(t)$  is non-increasing. We may thus define  $a = \lim_{t \rightarrow \infty} x(t)$ . Using the continuity of  $\psi$  yields to

$$\lambda e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty \psi(x(s)) e^{-s} ds dx = \lambda e^{-\lambda t} \int_0^t e^{\lambda x} \psi(a)(1 + o(1)) dx,$$

This last integral being divergent as  $t \rightarrow \infty$ , we deduce that

$$\lambda e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty \psi(x(s)) e^{-s} ds dx = \psi(a) + o(1).$$

From (4), we get that  $a = \psi(a)$  which implies that  $a \in \{\rho, 1\}$ . Note however that the inequality  $a < 1$  follows from Theorem 1 which implies that  $q(\lambda) < 1$ .  $\blacksquare$

From now on in this section, we fix a small  $u > 0$  and we assume that

$$\lambda_1 < \lambda < 1 - u. \quad (7)$$

We define the map  $A : \mathcal{H} \rightarrow L^\infty(\mathbb{R}_+, \mathbb{R}_+)$  defined by

$$A(y)(t) = e^{-\lambda t} + \lambda e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty \psi(y(s)) e^{-s} ds dx. \quad (8)$$

Since

$$\|\psi\|_{L^\infty([0,1], \mathbb{R}_+)} = \max_{x \in [0,1]} |\psi(x)| = 1$$

it is indeed straightforward to check that  $A(y)$  is in  $L^\infty(\mathbb{R}_+, \mathbb{R}_+)$ :  $A(y)(t)$  is bounded by 1. Note also that  $y \equiv 1$  is a solution of the fixed point equation

$$y = A(y).$$

By (4), we find that the function  $x$  defined by (3) satisfies also the fixed point  $x = A(x)$ . In the sequel, we are going to analyze the non trivial fixed points of  $A$ .

Let  $x \in \mathcal{H}$  such that  $x = Ax$ . Then  $x \not\equiv 1$ . By induction, it follows easily that  $t \mapsto x(t)$  is twice differentiable. In particular, from the above argument,  $x$  satisfies (6) and we are looking for a specific non-negative solution of (6) with  $x(0) = 1$ . To characterize completely this solution, it would be enough to compute  $x'(0)$  (which is necessary negative since  $x(0) = 1$ ,  $x'(0) = 0$  corresponds to the trivial solution  $x \equiv 1$ ). We will perform this in the next subsection in an asymptotic regime. We first gives some basic properties which follow from the phase diagram of the ODE (6).

Notice that if at some time  $t > 0$ ,  $x'(t) = 0$  then from (6) and  $\rho < x(t) < 1$ , we deduce that  $x''(t) > 0$ . In particular,  $x'(s) > 0$  for all  $s \in (t, t + \delta)$  for some  $\delta > 0$ . This contradicts that  $x(\cdot)$  is non-increasing. Also, from (5) and  $\lambda < 1$ , we find that for all  $t \geq 0$ ,

$$-1 < x'(t) < 0. \quad (9)$$

We define  $X(t) = (x(t), x'(t))$  and

$$F(x_1, x_2) = (x_2, (1 - \lambda)x_2 - \varphi(x_1))$$

so that

$$X' = F(X). \quad (10)$$

We define the trajectory  $\Phi = \{X(t) : t \geq 0\}$ . Recall that  $\rho = \lim_{t \rightarrow \infty} x(t)$ . Also, since for all  $t \geq 0$ ,  $X(t)_1' = F(X(t))_1 < 0$ ,  $\Phi$  is the graph of a differentiable function  $f : (\rho, 1] \rightarrow (-1, 0)$  with  $f(1) = x'(0) < 0$ ,

$$\Phi = \{(s, f(s)) : s \in (\rho, 1]\}.$$

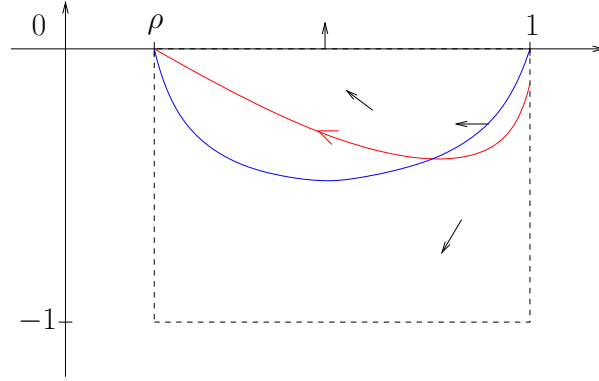


Figure 3: Illustration of the phase portrait. In blue, the curve  $\Gamma$ , in red the curve  $\Phi$ .

Moreover

$$f'(s) = \frac{F((s, f(s)))_2}{F((s, f(s)))_1} = 1 - \lambda - \frac{\varphi(s)}{f(s)}. \quad (11)$$

We notice that on the curve

$$\Gamma = \{(x_1, x_2) \in [\rho, 1] \times [-1, 0] : (1 - \lambda)x_2 = \varphi(x_1)\}$$

the second coordinate of  $F$  vanishes (see figure 3). The next lemma shows that our function  $(x(t), x'(t))$  cannot cross  $\Gamma$  near its origin  $(x(0), x'(0))$ .

**Lemma 11.** *There exists  $\delta > 0$  such that for any  $\lambda$  as in (7) the following holds. Let  $x \in \mathcal{H}$  such that  $x = Ax$ . If  $(x(t), x'(t)) \in \Gamma$  for some  $t > 0$ , then  $|x'(t)| > \delta$  and  $x(t) < 1 - \delta$ .*

*Proof:* Consider the larger  $s$  such that  $(s, f(s)) \in \Gamma$ . Recall that  $f(1) = x'(0) < 0$ . Thus, on  $(s, 1]$ ,  $(s, f(s))$  is below  $\Gamma$  and it follows that  $f$  is increasing.

We define  $\alpha \in (\rho, 1)$  as the point where the function  $\varphi(x)$  reaches its minimum. Since  $\varphi$  is decreasing on  $[\rho, \alpha]$ , it follows that  $s \in [\alpha, 1]$ . We set  $f(s) = -\delta$  and assume that  $0 < \delta < |\varphi(\alpha)|/(1 - \lambda)$ , so that  $s \in (\alpha, 1)$ . We will prove by contradiction that  $\delta$  cannot be arbitrarily small. This will conclude the proof of the lemma. Indeed, by construction  $(1 - \lambda)\delta = -\varphi(s)$  and  $\varphi$  has a continuous inverse in a neighborhood of 1 : as  $\delta$  goes to 0,  $s = s(\delta)$  goes to 1.

To this end, we fix any  $\beta \in (\alpha, 1)$  and set  $b = -\varphi(\beta) > 0$ . Assume that  $\delta$  is small enough so that  $\beta < s$ . Consider the solution  $Y(t) = (y(t), y'(t))$  of the ODE (10) with initial condition  $Y(0) = (\beta, -\delta)$ . The trajectory of  $Y(t)$  is denoted by  $\tilde{\Phi} = \{Y(t) : t \geq 0\}$ . We define the set  $\Gamma_+ = \{(x_1, x_2) \in [\alpha, 1] \times [-\delta, 0] : (1 - \lambda)x_2 \geq \varphi(x_1)\}$ . On  $\Gamma_+$ ,  $F(x)_1 < 0$  and  $F(x)_2 \geq 0$ . It follows that the trajectories  $\Phi$  and  $\tilde{\Phi}$  exit  $\Gamma_+$  either on its left side  $\{(\alpha, x_2), x_2 \in [-\delta, 0]\}$  or its upper side  $\{(x_1, 0), x_1 \in [\alpha, 1]\}$ . From (9), we deduce that  $\Phi$  exit  $\Gamma_+$  on the left side. Since  $\Phi$  and  $\tilde{\Phi}$  cannot intersect and  $\tilde{\Phi}$  is on the left side of  $\Phi$  in  $\Gamma_+$ , we deduce that necessarily,  $\tilde{\Phi}$  also exits  $\Gamma_+$  on the left side. We now check that if  $\delta$  is too small, this is not possible.

Define  $\tau > 0$  as the exit time of  $Y(t)$  from  $\Gamma_+$ . If  $0 \leq t \leq \tau$ , using that  $\varphi$  is increasing on  $[\alpha, \beta]$ , we find  $y''(t) \geq -(1 - \lambda)\delta + b \geq b/2$  for  $\delta$  small enough. We deduce that for all  $t \in [0, \tau]$ ,  $y'(t) \geq (b/2)t - \delta$  and  $y(t) \geq (b/4)t^2 - \delta t + \beta$ . Set  $t_e = 2\delta/b$ . We deduce that  $\tau \leq t_e$  and, if  $\delta$  is small enough,  $y(\tau) \geq (b/4)t_e^2 - \delta t_e + \beta = -\delta^2/b + \beta > \alpha$ . Hence, if  $\delta$  is too small,  $\tilde{\Phi}$  must exit  $\Gamma_+$  on the upper side. This is not possible.  $\blacksquare$

### 3.3 Comparison of second order differential equations

It is possible to compare the trajectories of solutions of second order ODE by using the phase diagram. For two functions  $\varphi_1, \varphi_2$  on  $[0, 1]$ , we write  $\varphi_1 \leq \varphi_2$  if for all  $t \in [0, 1]$ ,  $\varphi_1(t) \leq \varphi_2(t)$ .

**Lemma 12.** *Let  $\delta > 0$  be as in Lemma 11. Let  $x \in \mathcal{H}$  such that  $x = Ax$ . Let  $\tilde{\varphi}$  be a Lipschitz-continuous function and  $y$  be solution of  $y'' - (1 - \lambda)y' + \tilde{\varphi}(y) = 0$  with  $y(0) = 1$ ,  $y'(0) < 0$ . We define the exit times*

$$T = \inf\{t \geq 0 : (y(t), y'(t)) \notin (0, 1] \times (-1, 0)\},$$

$$T_- = \inf\{t \geq 0 : y'(t) \leq -1\} \quad \text{and} \quad T_+ = \inf\{t \geq 0 : (1 - \lambda)y'(t) = \varphi(y(t)), y(t) \geq 1 - \delta\}.$$

(i) *If  $T_+ = T < \infty$  and  $\tilde{\varphi} \geq \varphi$  then  $y'(0) \geq x'(0)$ .*

(ii) *If  $T_- = T < \infty$  and  $\tilde{\varphi} \leq \varphi$  then  $x'(0) \geq y'(0)$ .*

*Proof:* Let us start with the hypothesis of (i). The proof is by contradiction : we assume that  $y'(0) < x'(0)$ . We set  $Y(t) = (y(t), y'(t))$  and  $G(y_1, y_2) = (y_2, (1 - \lambda)y_2 - \tilde{\varphi}(y_1))$ . Define the trajectories  $\Phi = \{X(t) : t \geq 0\}$ , and for  $0 < \tau \leq T$ ,  $\tilde{\Phi}(\tau) = \{Y(t) : 0 \leq t < \tau\}$ . Recall that  $\Phi$  is the graph of an increasing function  $f$  given by (11).

Similarly, if  $y \in (0, 1] \times (-1, 0)$ ,  $G(y)_1 = y_2 < 0$ . Thus there exists a differentiable function  $g : (y(T), 1] \rightarrow (-1, 0)$  such that

$$\tilde{\Phi}(T) = \{(s, g(s)) : s \in (y(T), 1]\},$$

with

$$g'(s) = 1 - \lambda - \frac{\tilde{\varphi}(s)}{g(s)}.$$

Now, the assumption  $y'(0) < x'(0) < 0$  reads  $g(1) < f(1) < 0$ .

By assumption, there exists  $1 - \delta \leq s < 1$  such that  $(s, g(s)) \in \{(x_1, x_2) : (1 - \lambda)x_2 = \varphi(x_1), x_2 \leq 1 - \delta\}$ . Hence, by Lemma 11 and the intermediate value Theorem, there exists a largest  $s < s_1 < 1$  such that the curves intersect :  $g(s_1) = f(s_1)$  and  $g(s) < f(s)$  on  $(s_1, 1]$ .

Note however, that from (11), if  $s \in (s_1, 1]$ ,

$$g'(s) = 1 - \lambda - \frac{\tilde{\varphi}(s)}{g(s)} > 1 - \lambda - \frac{\varphi(s)}{f(s)} = f'(s).$$

Hence, integrating over  $(s_1, 1]$  we find  $g(1) - g(s_1) = g(1) - f(s_1) > f(1) - f(s_1)$ . This contradicts  $g(1) < f(1)$ . We have proved (i). The proof of (ii) is identical and is omitted. ■

### 3.4 Proof of Theorem 2

We first linearize (6) in the neighborhood of  $\lambda_1$ .

**Step one : linearization from below.** We have  $\varphi(1) = 0$ ,  $\varphi'(1) = \lambda(d-1) > 0$ , and from the convexity of  $\varphi$ ,

$$\varphi(s) \geq \lambda(d-1)(s-1). \quad (12)$$

We take  $\lambda_1 < \lambda < 1$  and consider the linearized differential equation

$$y'' - (1-\lambda)y' + \lambda(d-1)(y-1) = 0. \quad (13)$$

The solutions of this differential equation are

$$y(t) = 1 + a \sin(\omega t) e^{\frac{(1-\lambda)t}{2}} + b \cos(\omega t) e^{\frac{(1-\lambda)t}{2}},$$

where

$$\omega = \frac{1}{2} \sqrt{-\lambda^2 + 2(2d-1)\lambda - 1} = c(\lambda) \sqrt{\lambda - \lambda_1},$$

and

$$c(\lambda) = \frac{1}{2} \sqrt{(2d-1) + 2\sqrt{d(d-1)} - \lambda} = (d(d-1))^{1/4} + O(|\lambda - \lambda_1|).$$

We use this ODE to bound from below  $x'_\lambda(0)$ .

**Lemma 13.** *For all  $\lambda_1 < \lambda < 1$ ,*

$$x'_\lambda(0) \geq -e^{\frac{1}{1-\lambda}} e^{-\frac{\pi(1-\lambda)}{2\omega}} (1 + O(\omega^2)).$$

*Proof:* Let  $a < 0$ ,  $b = (1-\lambda)/2$ , and consider the function

$$y(t) = 1 + a \sin(\omega t) e^{bt}.$$

We have  $y(0) = 1$ ,  $y'(0) = a\omega$ ,

$$y'(t) = ae^{bt}(\omega \cos(\omega t) + b \sin(\omega t)),$$

and

$$y''(t) = ae^{bt}(2b\omega \cos(\omega t) + (b^2 - \omega^2) \sin(\omega t)).$$

Define

$$T = \frac{\pi}{\omega} - \frac{1}{\omega} \arctan\left(\frac{2b\omega}{b^2 - \omega^2}\right) = \frac{\pi}{\omega} - \frac{2}{b} + O(\omega^2).$$

On the interval  $(0, T)$ ,  $y''(t) < 0$  and  $y''(T) = 0$ . Thus the function  $y'(t)$  is decreasing on  $[0, T]$  and

$$y'(T) = e^{-\frac{2}{b}} ae^{\frac{\pi b}{\omega}} (\omega + O(\omega^3)).$$

Hence, we may choose  $a$  such that  $y'(T) = -1$  with

$$a = -\omega^{-1} e^{\frac{2}{b}} e^{-\frac{\pi b}{\omega}} (1 + O(\omega^2)).$$

It remains to use (12) with Lemma 12. ■

**Step two : linearization from above.** For  $0 < \eta < \min(1, c(\lambda)/(d-1))$ , we define

$$\ell = (1 - \eta)\lambda + \eta\lambda_1 < \lambda,$$

and the Lipschitz-continuous function

$$\tilde{\varphi}(s) = \max(\varphi(s), \ell(d-1)(s-1)).$$

In particular

$$\varphi \leq \tilde{\varphi}. \quad (14)$$

We define the linear differential equation

$$y'' - (1 - \lambda)y' + \ell(d-1)(y-1) = 0. \quad (15)$$

The solutions of (15) are

$$y(t) = 1 + a \sin(\omega' t) e^{\frac{(1-\lambda)t}{2}} + b \cos(\omega' t) e^{\frac{(1-\lambda)t}{2}},$$

with

$$\omega' = \frac{1}{2} \sqrt{-\lambda^2 + 2(2d-1)\lambda - 1 - 4\eta(d-1)(\lambda - \lambda_1)} = \omega \sqrt{1 - \frac{\eta(d-1)}{c(\lambda)}}.$$

A careful choice of  $a, \eta$  will lead to the following upper bound. In the sequel,  $o(1)$  denotes a function which goes to 0 as  $\omega$  goes to 0.

**Lemma 14.** *If  $\pi$  has finite second moment, then there exists a constant  $c_0 > 0$  such that for all  $\lambda_1 < \lambda < 1$ ,*

$$x'_\lambda(0) \leq -c_0 \omega^3 e^{-\frac{\pi(1-\lambda)}{2\omega}} (1 + o(1)).$$

*Proof:* We set

$$b = \frac{1 - \lambda}{2} \quad \text{and} \quad \kappa = \sqrt{1 - \frac{\eta(d-1)}{c(\lambda)}}.$$

We parametrize in terms of  $\kappa$ , so that

$$\ell = \lambda - (1 - \kappa^2)\omega^2 \quad \text{and} \quad \omega' = \kappa\omega. \quad (16)$$

For  $a < 0$ , we look at the solution

$$y(t) = 1 + a \sin(\omega\kappa t) e^{bt}.$$

We have  $y(0) = 1$ ,  $y'(0) = a\kappa\omega$ .

$$y'(t) = a e^{bt} (\omega\kappa \cos(\omega\kappa t) + b \sin(\omega\kappa t)).$$

We repeat the argument of Lemma 13. On the interval  $[0, T]$ ,  $y''(t) \geq 0$  and  $y''(T) = 0$ , where

$$T = \frac{\pi}{\omega\kappa} - \frac{1}{\omega\kappa} \arctan\left(\frac{\omega\kappa}{b^2 - \omega^2\kappa^2}\right) = \frac{\pi}{\omega\kappa} - \frac{2}{b} + O(\omega^2),$$



and the  $O(\cdot)$  is uniform over all  $\kappa > 1/2$ . The function  $y'(t)$  is increasing on  $[0, T]$  and

$$y'(T) = e^{-2} a e^{\frac{\pi b}{\omega \kappa}} (\omega \kappa + O(\omega^3)).$$

Now, we have  $\varphi(s) < \ell(d-1)(s-1)$  for all  $s \in [1-\sigma, 1]$  with

$$-\ell(d-1)\sigma = \varphi(1-\sigma) = \lambda(\psi(1-\sigma) - 1 + \sigma).$$

If  $\pi$  has finite second moment then, from Abel's Theorem,  $\psi''$  is continuous on  $[0, 1]$ . Also from Jensen's inequality,  $\psi''(1) \geq d(d-1) > 0$ . We expand  $\psi$  in a neighborhood of 1, as  $\omega \rightarrow 0$ , it yields to,

$$\begin{aligned} \sigma &= \frac{2(d-1)}{\psi''(1)\lambda} (\lambda - \ell)(1 + o(1)) \\ &= \frac{2(d-1)}{\psi''(1)\lambda} (1 - \kappa^2)\omega^2(1 + o(1)), \end{aligned}$$

where  $o(1)$  is uniform over all  $0 < \kappa < 1$ . In particular, for all  $\omega$  small enough,  $\sigma < \delta$  with  $\delta$  as in Lemma 11. Also, from (15), for  $t = T$ , since  $y''(T) = 0$ , we have

$$\frac{y(T) - 1}{y'(T)} = \frac{1 - \lambda}{\ell(d-1)} = \frac{2b}{\ell(d-1)}.$$

We may choose  $a$  such that  $y(T) = 1 - \sigma$  by setting

$$a = -\sigma e^2 \frac{\ell(d-1)}{2b} \frac{e^{-\frac{\pi b}{\omega \kappa}}}{\omega \kappa} (1 + O(\omega^2)) = -e^2 \frac{\ell(d-1)^2}{\lambda \psi''(1)b} e^{-\frac{\pi b}{\omega \kappa}} \frac{(1 - \kappa^2)\omega}{\kappa} (1 + o(1)).$$

By construction, with this choice of  $a$ , we have  $(1 - \lambda)y'(T) = \varphi(y(T))$ . Now, in the domain  $1 - \sigma \leq y \leq 1$  the non-linear differential equation  $y'' - (1 - \lambda)y' + \tilde{\varphi}(y)$  obviously coincides with (15). Thus, using (14) and Lemma 12, it yields to

$$x'(0) \leq y'(0) = -e^2 \frac{\ell(d-1)^2}{\lambda \psi''(1)b} e^{-\frac{\pi b}{\omega \kappa}} (1 - \kappa^2)\omega^2(1 + o(1)).$$

Taking finally  $\kappa = 1 - \omega/(\pi b)$  and using (16) give the statement. ■

**Step three : end of proof.** We may now complete the proof of Theorem 2. We start with the left hand side inequality. We first note that, by Lemma 11,  $x'(t)$  is decreasing on the interval  $[0, t_0]$  where  $t_0$  is the time where  $(x(t_0), x'(t_0)) \in \Gamma = \{(x_1, x_2) \in [\rho, 1] \times [-1, 0] : (1 - \lambda)x_2 = \varphi(x_1)\}$ . Moreover by Lemma 11, we find  $x(t_0) \leq 1 - \delta$ . However, by (9), we have

$$x(t) \geq 1 - t.$$

Hence  $t_0 \geq \delta$ . Then, by construction, on the interval  $[0, t_0]$ ,

$$x(t) \leq 1 + x'(0)t = 1 - |x'(0)|t.$$

Since  $\psi(x) \leq x$  on  $[\rho, 1]$ , it follows from (2) that the survival probability may be lower bounded as

$$\begin{aligned} 1 - q(\lambda) &= \int_0^\infty (1 - \psi(x(t)))e^{-t} dt \geq \int_0^\infty (1 - x(t))e^{-t} dt \\ &\geq \int_0^{t_0} |x'(0)|te^{-t} dt \\ &\geq |x'(0)| \int_0^\delta te^{-t} dt. \end{aligned}$$

It remains to use Lemma 14 and we obtain the left hand side of Theorem 2.

We turn to the right hand side inequality. For  $X = (x_1, x_2) \in [\rho, 1] \times (-\infty, 0)$ , define  $G(X) = (x_2, (1 - \lambda)x_2)$ . From the definition of  $F$  in (10), we have, component-wise, for any  $X \in [\rho, 1] \times (-\infty, 0)$ ,

$$F(X) \geq G(X).$$

Note also that  $G$  is monotone : if component-wise  $X \geq Y$  then  $G(X) \geq G(Y)$ . It follows easily that if  $X(0) = Y(0)$ ,  $X' = F(X)$  and  $Y' = G(Y)$  then component-wise

$$X(t) \geq Y(t).$$

Looking at the solution of  $y'' - (1 - \lambda)y' = 0$  such that  $y(0) = 1$  and  $y'(0) = x'(0)$ , we get that

$$x(t) \geq 1 + x'(0)(e^{(1-\lambda)t} - 1).$$

We deduce from (2)-(3) and the convexity of  $\psi$  that,

$$\begin{aligned} q(\lambda) &= \int_0^\infty \psi(x(t))e^{-t} dt \\ &\geq \int_0^\infty \psi(1 + x'(0)(e^{(1-\lambda)t} - 1))e^{-t} dt \\ &\geq \int_0^\infty \left(1 + dx'(0)(e^{(1-\lambda)t} - 1)\right)e^{-t} dt \\ &\geq 1 + dx'(0)/\lambda. \end{aligned}$$

We finally apply Lemma 13 and this concludes the proof of Theorem 2. ■

## 4 Proofs of Theorems 3-4

### 4.1 Proof of Theorem 4

For the tree  $T^\downarrow$ , without counting  $o$ , let  $Y(t)$  be the total number of recovered vertices when the process reach its absorbing state given that the root dies at time  $t$ . By definition, if  $D$  is an independent exponential variable with mean 1, then

$$Z \stackrel{d}{=} Y(D),$$

where the symbol  $\stackrel{d}{=}$  stands for distributional equality.

In  $T$ , we denote the offsprings of the root by  $\{1, \dots, N\}$ . The random variable  $N$  has distribution  $\pi$ . The root infects each of its offspring after an independent exponential variable with intensity  $\lambda$ . Note that in  $T$ , the subtrees generated by each of the offsprings of the root are iid copies of  $T$ . Hence, the recursive structure of the tree  $T$  leads to the following equality in distribution

$$Y(t) \stackrel{d}{=} 1 + \sum_{i=1}^N \mathbf{1}(\xi_i \leq t) Y_i(t - \xi_i + D_i). \quad (17)$$

where  $(\xi_i)_{i \in \mathbb{N}}$  are iid exponential variables with intensity  $\lambda$ ,  $(Y_i)_{1 \leq i \leq N}$  and  $(D_i)_{1 \leq i \leq N}$  are independent copies of  $Y$  and  $D$  respectively. Note that since all variables are non-negative, there is no issue with the case  $Y(t) = +\infty$  in the above recursive distributional equation (RDE). The RDE (17) is the cornerstone of the argument.

We start with a lemma

**Lemma 15.** *Let  $t > 0$  and  $u \geq 1$ , if  $\mathbb{E}'_\lambda[Z^u] < \infty$  then  $\mathbb{E}'_\lambda[Y(t)^u] < \infty$ .*

*Proof:* Since  $Z \stackrel{d}{=} Y(D)$ , from Fubini's Theorem,  $\mathbb{E}'_\lambda[Z^u] = \int_0^\infty \mathbb{E}'_\lambda[Y(t)^u] e^{-t} dt$ . Therefore  $\mathbb{E}'_\lambda[Y(t)^u] < \infty$  for almost all  $t \geq 0$ . Note however that since  $t \mapsto Y(t)$  is monotone for the stochastic domination, it implies that  $\mathbb{E}'_\lambda[Y(t)^u] < \infty$  for all  $t \geq 0$ . ■

Now, assume that  $\mathbb{E}'_\lambda Z < \infty$ . We may then take expectation in (17):

$$\mathbb{E}'_\lambda Y(t) = 1 + d \int_0^t \int_0^\infty \mathbb{E}'_\lambda[Y(t - x + s)] e^{-s} ds \lambda e^{-\lambda x} dx.$$

Let  $f_1(t) = \mathbb{E}'_\lambda Y(t)$ , it satisfies the integral equation, for all  $t \geq 0$ ,

$$f_1(t) = 1 + \lambda d e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_1(s) e^{-s} ds dx. \quad (18)$$

Multiplying by  $e^{\lambda t}$  and taking derivative, we get:

$$(f_1'(t) + \lambda f_1(t)) e^{\lambda t} = \lambda e^{\lambda t} + \lambda d e^{(\lambda+1)t} \int_t^\infty f_1(s) e^{-s} ds.$$

Then, multiplying by  $e^{-(\lambda+1)t}$ , taking the derivative a second time and then re-multiplying by  $e^t$ , we obtain:  $f_1''(t) - (1 - \lambda)f_1'(t) - \lambda f_1 = -\lambda - \lambda d f_1(t)$ . So, finally,  $f_1$  solves a linear ordinary differential equation of the second order

$$x'' - (1 - \lambda)x' + \lambda(d - 1)x = -\lambda,$$

with initial condition  $f_1(0) = 1$ . We get that

$$f_1(t) = x(t) - \frac{1}{d - 1},$$

where  $x(t)$  solves the ordinary differential equation

$$x'' - (1 - \lambda)x' + \lambda(d - 1)x = 0. \quad (19)$$

with  $x(0) = d/(d-1)$ . The discriminant of the polynomial  $X^2 - (1-\lambda)X + \lambda(d-1) = 0$  is

$$\Delta = \lambda^2 - 2\lambda(2d-1) + 1.$$

If  $0 < \lambda < \lambda_1$ , the discriminant is positive. The roots of the polynomial are

$$\alpha = \frac{1-\lambda-\sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{1-\lambda+\sqrt{\Delta}}{2}. \quad (20)$$

The solutions of (19) are

$$x(t) = \frac{d}{d-1} \left( (1-a)e^{\alpha t} + ae^{\beta t} \right) \quad (21)$$

for some constant  $a$ .

Similarly, if  $\lambda = \lambda_1$ , then  $\alpha = \beta = 1-d+\sqrt{d(d-1)}$  and the solutions of (19) are

$$x(t) = \frac{d}{d-1} (at+1)e^{\alpha t}. \quad (22)$$

For  $0 < \lambda \leq \lambda_1$ , we check easily that the functions  $x(\cdot)$  with  $a \geq 0$  are the nonnegative solutions of the integral equation (18).

It remains to prove that if  $0 < \lambda \leq \lambda_1$  then  $\mathbb{E}Z < \infty$  and

$$f_1(t) = \frac{de^{\alpha t} - 1}{d-1}.$$

Indeed, we would get  $\mathbb{E}Z = \int_0^\infty \mathbb{E}Y(t)e^{-t}dt = \frac{d}{d-1} \frac{1}{1-\alpha} - \frac{1}{d-1}$  as stated in Theorem 4. To this end, we consider  $T_n$  as being a Galton-Watson tree with offspring distribution  $\pi$  stopped at generation  $n$ . We denote by  $Y^{(n)}(t)$  the total number of recovered particles given that the root dies at time  $t$ . We have  $Y^{(0)}(t) = 1$  and for all  $n \geq 0$ , as in RDE (17),

$$Y^{(n+1)}(t) \stackrel{d}{=} 1 + \sum_{i=1}^N \mathbf{1}(\xi_i \leq t) Y_i^{(n)}(t - \xi_i + D_i),$$

where  $Y_i^{(n)}$ , and  $D_i$  are independent copies of  $Y^{(n)}$  and  $D$  respectively. Since  $\mathbb{E}N < \infty$ , for all  $n \geq 0$ ,  $f_1^{(n)}(t) = \mathbb{E}Y^{(n)}(t) < \infty$ . Taking expectation, we have for all  $t \geq 0$ ,

$$f_1^{(n+1)}(t) = 1 + \lambda de^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_1^{(n)}(s)e^{-s} ds dx = \Phi(f_1^{(n)})(t), \quad (23)$$

where  $\Phi$  is the mapping

$$\Phi : g \mapsto 1 + \lambda de^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty g(s)e^{-s} ds dx.$$

It is easy to check that  $\Phi$  is indeed a mapping from  $\mathcal{H}_1$  to  $\mathcal{H}_1$ , where  $\mathcal{H}_1$  is the set of non-decreasing functions  $g : [0, \infty) \rightarrow [1, \infty)$  such that  $\sup_{t \geq 0} g(t)e^{-\alpha t} < \infty$ . Now, from what precedes the function

$$h(t) = \frac{de^{\alpha t} - 1}{d-1}.$$

is a fixed point of  $\Phi$ .

We notice that  $f_1^{(0)}(t) \leq h(t)$ . Due to the monotony of the mapping  $\Phi$  we get  $f_1^{(1)} = \Phi(f_1^{(0)}) \leq \Phi(h) = h$ . By recursion, it follows  $f_1^{(n)} \leq h$ . The monotone convergence Theorem implies that  $f_1(t) = \lim_{n \rightarrow \infty} f_1^{(n)}(t)$  exists and is bounded by  $h(t)$ . Therefore  $f_1$  solves the integral equation (18) and is equal to  $x - 1/(d-1)$  where  $x$  is given by (21) (or (22) if  $\lambda = \lambda_1$ ) for some  $a \geq 0$ . However, from what precedes, we get  $x(t) \leq h(t) + 1/(d-1)$  and the only possibility is  $a = 0$  and  $f_1(t) = h(t)$ .

This concludes the proof of Theorem 4. ■

## 4.2 Proof of Theorem 3 for integer moments

For  $0 < \lambda < \lambda_1$ , we define

$$\bar{\gamma} = \frac{\lambda^2 - 2d\lambda + 1 - (1 - \lambda)\sqrt{\Delta}}{2\lambda(d-1)} = \frac{\beta}{\alpha}. \quad (24)$$

The key property of  $\bar{\gamma}(\lambda)$  is that  $(1 - \lambda)u\alpha - \lambda(d-1) - u^2\alpha^2 > 0$  if and only if  $1 < u < \bar{\gamma}(\lambda)$ . We also note that if  $u > 1$ ,  $u < \bar{\gamma}$  is equivalent to  $\lambda \in (0, \lambda_u)$ . We first state an important lemma. Let  $1 < u < \bar{\gamma}$ , we define  $\mathcal{H}_u$ , the set of measurable functions  $h : [0, \infty) \rightarrow [0, \infty)$  such that  $\sup_{t \geq 0} h(t)e^{-u\alpha t} < \infty$ . Let  $C > 0$ , we define the mapping from  $\mathcal{H}_u$  to  $\mathcal{H}_u$ ,

$$\Psi : h \mapsto Ce^{u\alpha t} + \lambda de^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty h(s)e^{-s} ds dx.$$

In order to check that  $\Psi$  is indeed a mapping from  $\mathcal{H}_u$  to  $\mathcal{H}_u$ , we use the fact that if  $1 < u < \bar{\gamma} = \beta/\alpha$  then  $u\alpha < \beta < 1$ .

**Lemma 16.** *Let  $1 < u < \bar{\gamma}$  and  $f \in \mathcal{H}_u$  such that  $f \leq \Psi(f)$ . Then for all  $t \geq 0$ ,*

$$f(t) \leq C \frac{(u\alpha + \lambda)(1 - u\alpha)e^{u\alpha t}}{(1 - \lambda)u\alpha - \lambda(d-1) - u^2\alpha^2}.$$

*Proof:* We set  $g_0 = f$  and for  $k \geq 1$ , we define  $g_k = \Psi(g_{k-1})$ . First, since  $1 < u < \bar{\gamma}$  then  $(u\alpha + \lambda)(1 - u\alpha) > \lambda d$ . We use the formula for all  $u \geq 0$  such that  $u\alpha < 1$ :

$$\lambda e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty e^{\alpha u s} e^{-s} ds dx = \frac{\lambda(e^{\alpha u t} - e^{-\lambda t})}{(u\alpha + \lambda)(1 - u\alpha)}. \quad (25)$$

We deduce easily that if  $g_0(t) \leq Le^{u\alpha t}$  then

$$g_1(t) = \Psi(g_0)(t) \leq Ce^{u\alpha t} + \frac{L\lambda d}{(u\alpha + \lambda)(1 - u\alpha)}(e^{u\alpha t} - e^{-\lambda t}) \leq L_1 e^{u\alpha t},$$

with  $L_1 = C + \frac{L\lambda d}{(u\alpha + \lambda)(1 - u\alpha)}$ . By recursion, we obtain that  $\limsup_k g_k(t) \leq L_\infty e^{u\alpha t}$ , with  $L_\infty = C(u\alpha + \lambda)(1 - u\alpha)/((1 - \lambda)u\alpha - \lambda(d-1) - u^2\alpha^2) < \infty$ .

We may now conclude the proof. Notice that  $\Psi$  is monotone: if for all  $t \geq 0$ ,  $h_1(t) \geq h_2(t)$  then for all  $t \geq 0$ ,  $\Psi(h_1)(t) \geq \Psi(h_2)(t)$ . Hence, by recursion, from the assumption  $f \leq \Psi(f) = g_1$ , we deduce that for all integer  $k \geq 1$ ,  $f \leq g_k$ . It remains to take the limit in  $k$ . ■

Now, let  $p$  be an integer, and define  $f_p(t) = \mathbb{E}'_\lambda[Y(t)^p]$ . The main result of this subsection is the following lemma.

**Lemma 17.** *Let  $1 \leq p < \gamma_\pi$ , if  $\lambda \in (0, \lambda_p)$ , then  $f_p$  is finite and there exists a constant  $C_p$  such that for all  $t > 0$*

$$f_p(t) \leq C_p e^{p\alpha t}.$$

*Proof:* In §4.1, we have computed  $f_p$  for  $p = 1$  and found  $f_1(t) = (de^{\alpha t} - 1)/(d - 1)$ . Let  $p \geq 2$  and assume now that the statement of Lemma 17 holds for  $q = 1, \dots, p - 1$ . Let  $\kappa > 0$ ,  $Y^{(\kappa)}(t) = \min(Y(t), \kappa)$  and let  $\leq_{st}$  denote the stochastic domination, from RDE (17), we have

$$Y^{(\kappa)}(t) \leq_{st} 1 + \sum_{i=1}^N \mathbf{1}(\xi_i \leq t) Y_i^{(\kappa)}(t - \xi_i + D_i). \quad (26)$$

Recall the multinomial formula

$$\left( \sum_{i=1}^n y_i \right)^p = \sum_{p_1, \dots, p_n} \binom{n}{p_1 \dots p_n} y_1^{p_1} \dots y_n^{p_n}.$$

where the summation is taken over  $n$ -tuples of integers that sum up to  $p$ . Taking power  $p$  in the above stochastic inequality and expanding brutally, we thus get

$$Y^{(\kappa)}(t)^p \leq_{st} \sum_{p_1, \dots, p_{N+1}} \binom{N+1}{p_1 \dots p_{N+1}} \prod_{i=1}^N \left( \mathbf{1}_{p_i=0} + \mathbf{1}_{p_i \geq 1} \mathbf{1}(\xi_i \leq t) Y_i^{(\kappa)}(t - \xi_i + D_i)^{p_i} \right),$$

where the summation is taken over  $N + 1$ -tuples of integers that sum up to  $p$ . Now we define

$$f_p^{(\kappa)}(t) = \mathbb{E}'_\lambda \left[ Y^{(\kappa)}(t)^p \right] = \mathbb{E}'_\lambda [\min(Y(t), \kappa)^p].$$

Taking expectation and using independence leads to

$$\begin{aligned} f_p^{(\kappa)}(t) &\leq \sum_{n=0}^{\infty} \pi(n) \sum_{p_1, \dots, p_{n+1}} \binom{n+1}{p_1 \dots p_{n+1}} \prod_{i=1}^n \left( \mathbf{1}_{p_i=0} + \mathbf{1}_{p_i \geq 1} \mathbb{E}'_\lambda \left[ \mathbf{1}(\xi \leq t) Y^{(\kappa)}(t - \xi + D)^{p_i} \right] \right) \\ &\leq \sum_{n=0}^{\infty} \pi(n) \sum_{p_1, \dots, p_{n+1}} \binom{n+1}{p_1 \dots p_{n+1}} \\ &\quad \times \prod_{i=1}^n \left( \mathbf{1}_{p_i=0} + \mathbf{1}_{p_i \geq 1} \lambda e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_{p_i}^{(\kappa)}(s) e^{-s} ds dx \right). \end{aligned} \quad (27)$$

Consider a  $n + 1$ -tuple that sums up to  $p$  such that for all  $i = 1, \dots, n$ ,  $p_i \leq p - 1$ ,  $\sum_{i=1}^n p_i = q \leq p$  and  $\sum_{i=1}^n \mathbf{1}_{p_i \geq 1} = m \leq p$ . From the recursive hypothesis and (25), with  $L_1 = \max_{1 \leq k \leq p-1} C_k$ , we get

$$\begin{aligned} \prod_{i=1}^n \left( \mathbf{1}_{p_i=0} + \mathbf{1}_{p_i \geq 1} \lambda e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_{p_i}^{(\kappa)}(s) e^{-s} ds dx \right) &\leq \prod_{i: p_i \geq 1} \frac{C_{p_i} \lambda e^{\alpha p_i t}}{(\lambda + p_i \alpha)(1 - p_i \alpha)} \\ &\leq L_1^m e^{\alpha q t} e^{-\sum_{i=1}^n \ln(1 - p_i \alpha)}. \end{aligned}$$

Now recall that  $|\ln(1-y) + y| \leq \frac{y^2}{2(1-y)}$  for  $y \in (0, 1)$ . Since  $\sum_{i=1}^n p_i^2 \leq q^2$ , we get

$$\begin{aligned} \prod_{i=1}^n \left( \mathbf{1}_{p_i=0} + \mathbf{1}_{p_i \geq 1} \lambda e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_{p_i}(s) e^{-s} ds dx \right) &\leq L_1^m e^{\alpha q t} e^{\alpha q + \frac{\alpha^2 q^2}{2(1-p\alpha)}} \\ &\leq L_2 e^{\alpha p t}. \end{aligned}$$

Then, grouping together all such  $n+1$ -tuples, from (27) we deduce

$$\begin{aligned} f_p^{(\kappa)}(t) &\leq \sum_{n=0}^{\infty} \pi(n) \left( (n+1)^p L_2 e^{\alpha p t} + n \lambda e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_p^{(\kappa)}(s) e^{-s} ds dx \right) \\ &\leq L_2 e^{\alpha p t} + \lambda d e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_p^{(\kappa)}(s) e^{-s} ds dx, \end{aligned} \quad (28)$$

where we have used the hypothesis  $p < \gamma_\pi$ . We may then apply Lemma 16: there exists a constant  $C_p$  such that

$$f_1^{(\kappa)}(t) \leq C_p e^{\alpha p t}.$$

The monotone convergence Theorem implies that  $f_p(t) = \lim_{n \rightarrow \infty} f_1^{(\kappa)}(t)$  exists and is bounded by  $C_p e^{\alpha p t}$ . The recursion is complete.  $\blacksquare$

### 4.3 Proof of Theorem 3 : lower bound on $\gamma(\lambda)$

To prove Theorem 3, we shall prove two statements

$$\text{If } \mathbb{E}'_\lambda[Z^u] < \infty \text{ then } u \leq \overline{\gamma}, \quad (29)$$

$$\text{If } 1 < u < \min(\overline{\gamma}, \gamma_\pi) \text{ then } \mathbb{E}'_\lambda[Z^u] < \infty. \quad (30)$$

In this paragraph, we prove (30). The argument is a refinement of the argument in §4.2. Let  $\kappa > 0$  and let  $f_u^{(\kappa)}(t) = \mathbb{E}'_\lambda[\min(Y(t), \kappa)^u]$ , we have the following lemma.

**Lemma 18.** *If  $1 < u < \min(\overline{\gamma}, \gamma_\pi)$ , there exists a constant  $C_u > 0$  such that for all  $t \geq 0$  and  $\kappa > 0$ ,*

$$f_u^{(\kappa)}(t) \leq C_u e^{u \alpha t} + \lambda d e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_u^{(\kappa)}(s) e^{-s} ds dx.$$

*Proof:* The lemma is already proved if  $u$  is an integer in (28). The general case extends of the same argument. We write  $u = p + v$  with  $v \in (0, 1)$  and integer  $p \geq 1$ . We use the inequality, for all  $y_i \geq 0$ ,  $1 \leq i \leq n$ ,

$$\left( \sum_{i=1}^n y_i \right)^u \leq \sum_{i=1}^n \sum_{p_1, \dots, p_n} \binom{n}{p_1 \dots p_n} y_i^{p_i+v} \prod_{1 \leq j \leq n, j \neq i} y_j^{p_j}, \quad (31)$$

where the summation is taken over  $n$ -tuples of integers that sum up to  $p$  (which follows from the inequality  $(\sum y_i)^v \leq \sum y_i^v$  and the multinomial formula). Then from (26), we get the

stochastic domination

$$Y^{(\kappa)}(t)^u \leq_{st} \sum_{i=1}^N \sum_{p_1, \dots, p_{N+1}} \binom{N+1}{p_1 \dots p_{N+1}} \left( \mathbf{1}_{p_i=0} + \mathbf{1}_{p_i \geq 1} \mathbf{1}(\xi_i \leq t) Y_i^{(\kappa)}(t - \xi_i + D_i)^{p_i+v} \right) \\ \times \prod_{1 \leq j \leq N, j \neq i} \left( \mathbf{1}_{p_j=0} + \mathbf{1}_{p_j \geq 1} \mathbf{1}(\xi_j \leq t) Y_j^{(\kappa)}(t - \xi_j + D_j)^{p_j} \right), \quad (32)$$

where the summation is taken over  $N+1$ -tuples of integers that sum up to  $p$ . From Lemma 17, there exists  $C$  such that for all  $1 \leq q \leq p$ ,  $f_q(t) \leq C_q e^{q\alpha t}$ . Note also, by Jensen inequality, that for all  $1 \leq q \leq p-1$ ,  $f_{q+v}(t) \leq f_p(t)^{\frac{q+v}{p}} \leq C_p e^{(q+v)\alpha t}$ . The same argument (with  $p$  replaced by  $u$ ) which led to (28) in the proof of Lemma 17 leads to the result.  $\blacksquare$

Statement (30) is a consequence of Lemma 16 and Lemma 18. Indeed, by Lemma 16, for all  $t \geq 0$ ,  $f_u^{(\kappa)}(t) \leq C_1 e^{u\alpha t}$  for some positive constant  $C_u$  independent of  $\kappa$ . From the monotone convergence Theorem, we deduce that, for all  $t \geq 0$ ,  $f_u(t) \leq C_u e^{u\alpha t}$ . However from  $Z \stackrel{d}{=} Y(D)$ , we find

$$\mathbb{E}'_\lambda Z^u = \int_0^\infty f_u(t) e^{-t} dt \leq \int_0^\infty C_u e^{u\alpha t} e^{-t} dt.$$

Then, statement (30) follows from  $u\alpha < 1$ .

#### 4.4 Proof of Theorem 3 : upper bound on $\gamma(\lambda)$

In this paragraph, we prove statement (29). This will conclude the proof of Theorem 3. Let  $u > 1$ , we assume that  $\mathbb{E}'_\lambda[Z^u] < \infty$  we need to show that  $\lambda < \lambda_u$ . Without loss of generality we can assume that  $\lambda < \lambda_1$ . From Lemma 15 and (17), we get

$$f_u(t) = \mathbb{E}'_\lambda[Y(t)^u] = \mathbb{E}'_\lambda \left( 1 + \sum_{i=1}^N \mathbf{1}(\xi_i \leq t) Y_i(t - \xi_i + D_i) \right)^u.$$

Taking expectation and using the inequality  $(x+y)^u \geq x^u + y^u$ , for all positive  $x$  and  $y$ , we get:

$$f_u(t) \geq 1 + \lambda d e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_u(s) e^{-s} ds dx. \quad (33)$$

From Jensen's Inequality,  $f_u(t) \geq f_1(t)^u \geq e^{u\alpha t}$ . Note that the integral  $\int_x^\infty e^{\alpha s} e^{-s} ds$  is finite if and only if  $u < \alpha^{-1}$ . Suppose now that  $\bar{\gamma} < u < \alpha^{-1}$ . We use the fact: if  $u > \bar{\gamma}$  then  $u^2\alpha^2 - (1-\lambda)u\alpha + \lambda(d-1) > 0$ , to deduce that there exist  $0 < \epsilon < \lambda$  such that

$$u^2\alpha^2 - (1-\lambda)u\alpha + \lambda(d-1) > \epsilon d. \quad (34)$$

We define  $\tilde{\lambda} = \lambda - \epsilon$ ,  $\tilde{\Delta}(\epsilon) = (1-\lambda)^2 - 4(\tilde{\lambda}d - \lambda)$ . Note that  $\tilde{\Delta}(0) = \Delta$ . Since  $\lambda < \lambda_1$ , for  $\epsilon$  small enough,  $\tilde{\Delta}$  is non-negative, we may then consider the real roots of  $X^2 - (1-\lambda)X + \tilde{\lambda}d - \lambda$ :

$$\tilde{\alpha}(\epsilon) = \frac{1-\lambda-\sqrt{\tilde{\Delta}}}{2} \quad \text{and} \quad \tilde{\beta}(\epsilon) = \frac{1-\lambda+\sqrt{\tilde{\Delta}}}{2}.$$



Again, for  $\epsilon = 0$ ,  $\tilde{\alpha}(0) = \alpha$  and  $\tilde{\beta}(0) = \beta$ . Hence, since  $u > \bar{\gamma} = \beta/\alpha$ , by continuity, for  $\epsilon$  small enough,

$$u\alpha > \tilde{\beta}. \quad (35)$$

We compute a lower bound from (33) as follows:

$$\begin{aligned} f_u(t) &\geq 1 + \epsilon d e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_u(s) e^{-s} ds dx + \tilde{\lambda} d e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_u(s) e^{-s} ds dx \\ &\geq 1 + \epsilon d e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty e^{u\alpha s} e^{-s} ds dx + \tilde{\lambda} d e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_u(s) e^{-s} ds dx \\ &\geq 1 + L(e^{u\alpha t} - e^{-\lambda t}) + \tilde{\lambda} d e^{-\lambda t} \int_0^t e^{(\lambda+1)x} \int_x^\infty f_u(s) e^{-s} ds dx, \end{aligned}$$

with

$$L = \frac{\epsilon d}{(\lambda + u\alpha)(1 - u\alpha)} > 0.$$

We consider the mapping  $\Psi : h \mapsto 1 + L(e^{u\alpha t} - e^{-\lambda t}) + \tilde{\lambda} \int_0^t e^x \int_x^\infty f_u(s) e^{-s} ds dx$ .  $\Psi$  is monotone: if for all  $t \geq 0$ ,  $h_1(t) \geq h_2(t)$  then for all  $t \geq 0$ ,  $\Psi(h_1)(t) \geq \Psi(h_2)(t)$ . Since, for all  $t \geq 0$ ,  $f_u(t) \geq \Psi(f_u)(t) \geq 1$ , we deduce by iteration that there exists a function  $h$  such that  $h = \Psi(h) \geq 1$ . As in §4.1, solving  $h = \Psi(h)$  is simple, taking twice the derivative, we get,

$$h'' - (1 - \lambda)h' + (\tilde{\lambda}d - \lambda)h = -\lambda - L(\lambda + u\alpha)(1 - u\alpha)e^{u\alpha t}.$$

Therefore,  $h = ae^{\tilde{\alpha}t} + be^{\tilde{\beta}t} - \epsilon(u^2\alpha^2 - (1 - \lambda)u\alpha + \lambda(d - 1))^{-1}e^{u\alpha t}$  for some constant  $a$  and  $b$ . From (35) the leading term as  $t$  goes to infinity is equal to  $-\epsilon(u^2\alpha^2 - (1 - \lambda)u\alpha + \lambda(d - 1))^{-1}e^{u\alpha t}$ . However from (34),  $-\epsilon(u^2\alpha^2 - (1 - \lambda)u\alpha + \lambda(d - 1))^{-1} < 0$  and it contradicts the assumption that  $h(t) \geq 1$  for all  $t \geq 0$ . Therefore we have proved that  $u \leq \bar{\gamma}$ . ■

## 5 Proof of Theorem 7

### 5.1 Differential equation for the survival probability

The proof of Theorem 7 is essentially identical to the proof of Theorem 2. We first determine the differential equation associated to the probability of extinction. Define  $Q_\lambda(t)$  to be the extinction probability given that the root dies at time  $t > 0$  so that

$$q(\lambda) = \int_0^\infty Q_\lambda(t) e^{-t} dt \quad (36)$$

and  $Q_\lambda(0) = 1$ . Let  $\{\xi_i\}_{i \geq 1}$  be the arrival times of  $\Xi_\phi$  with  $0 \leq \xi_1 \leq \xi_2 \leq \dots$ . For integer  $i$  with  $1 \leq \xi_i \leq D_\phi$ , we define  $\mathcal{B}_i$  as the subprocess on particles  $i\mathbb{N}^f$  with ancestors  $i$ . For the process  $\mathcal{B}$  to get extinct, all the processes  $\mathcal{B}_i$  must get extinct. Conditioned on  $\Xi_\phi$ , and on the root to die at time  $D_\phi = t$ , the evolutions of the  $(\mathcal{B}_i)$  then become independent. Moreover, on this conditioning,  $\mathcal{B}_i$  is a birth-and-assassination process conditioned on their root to be at risk at time  $t - \xi_i$ . Hence, we get

$$Q_\lambda(t) = \mathbb{E}_\lambda \left[ \prod_{\xi_i \leq t} Q_\lambda(t - \xi_i + D_i) \right] = \mathbb{E}_\lambda \left[ \prod_{\xi_i \leq t} Q_\lambda(\xi_i + D_i) \right],$$

where  $\{\xi_i\}_{i \geq 1}$  is a Poisson point process of intensity  $\lambda$  and  $(D_i), i \geq 1$ , independent exponential variables with parameter 1. Using Levy-Khinchin formula, we deduce

$$\begin{aligned} Q_\lambda(t) &= \exp \left( \lambda \int_0^t (\mathbb{E} Q_\lambda(x + D_1) - 1) dx \right) \\ &= \exp \left( \lambda \int_0^t \int_0^\infty (Q_\lambda(x + s) - 1) e^{-s} ds dx \right). \end{aligned}$$

So finally, for any  $t \geq 0$ ,

$$Q_\lambda(t) = \exp \left( -\lambda t + \lambda \int_0^t e^x \int_x^\infty Q_\lambda(s) e^{-s} ds dx \right). \quad (37)$$

Performing the change of variable

$$x(t) = -\ln Q_\lambda(t) \quad (38)$$

and differentiating (37) once yields

$$x'(t) = e^t \int_t^\infty \varphi(x(s)) e^{-s} ds, \quad (39)$$

where  $\varphi(y) = \lambda(1 - e^{-y})$ . Now, multiplying the above expression by  $e^{-t}$  and differentiating once again, we find that  $x(t)$  satisfies the differential equation

$$x'' - x' + \varphi(x) = 0. \quad (40)$$

## 5.2 A fixed point equation

Let  $\mathcal{D}$  be the set of increasing Lipschitz-continuous functions  $\varphi$  on  $\mathbb{R}_+$  such that  $\varphi(0) = 0$ .  $\mathcal{D}_0$  is the subset of functions in  $\mathcal{D}$  such that  $\|\varphi\|_\infty < \infty$ . Let  $\mathcal{H}$  be the set of measurable functions  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $f(0) = 0$  and for any  $a > 0$ ,

$$\lim_{s \rightarrow \infty} e^{-as} f(s) = 0.$$

For  $\varphi \in \mathcal{D}_0$ , we define the map  $T_\varphi : \mathcal{H} \rightarrow \mathcal{H}$  defined by

$$T_\varphi(y)(t) = \int_0^t e^x \int_x^\infty \varphi(y(s)) e^{-s} ds dx. \quad (41)$$

Since  $\|\varphi\|_\infty < \infty$ , it is indeed straightforward to check that  $T_\varphi(y)$  is indeed an element of  $\mathcal{H}$  ( $T_\varphi(y)(t)$  is bounded by  $\|\varphi\|_\infty t$ ). Note also that  $y \equiv 0$  is a solution of the fixed point equation

$$y = T_\varphi(y).$$

If  $\varphi(s) = \lambda(1 - e^{-s})$ , then using (37) we find that the function  $x$  defined by (38) satisfies also the fixed point  $x = T_\varphi(x)$ . In the sequel, we are going to analyze the non trivial fixed points of  $T_\varphi$  with  $\varphi \in \mathcal{D}$ .

Let  $x \in \mathcal{H}$  such that  $x = T_\varphi x$  and  $x \not\equiv 0$ . By induction, it follows easily that  $t \mapsto x(t)$  is twice differentiable. In particular, since  $x(s) \geq 0$ ,  $x'(t) \geq 0$  and the function  $x : \mathbb{R}_+ \rightarrow \mathbb{R}_+$

is non-decreasing. Moreover, by assumption there exists  $t_0 > 0$  such that  $x(t_0) > 0$ . By monotony, we deduce that  $x(t) > 0$  for all  $t > t_0$ . Then, using again (39), we find that for all  $t \geq 0$ ,

$$0 < x'(t) < \|\varphi\|_\infty. \quad (42)$$

From the above argument, we are looking for a specific non-negative solution of (40) which satisfies  $x(0) = 0$ . To characterizes completely this solution, it would be enough to compute  $x'(0)$  (which is necessary positive since  $x(0) = x'(0) = 0$  corresponds to the trivial solution  $x \equiv 0$ ). We first gives some basic properties. We define  $X(t) = (x(t), x'(t))$  so that

$$X' = F(X) \quad (43)$$

with  $F((x_1, x_2)) = (x_2, x_2 - \varphi(x_1))$ . We also introduce the set

$$\Delta = \{(x_1, x_2) \in \mathbb{R}_+^2 : \varphi(x_1) < x_2 < \|\varphi\|_\infty\}.$$

**Lemma 19.** *Let  $\varphi \in \mathcal{D}_0$  and  $x \in \mathcal{H}$  such that  $x = T_\varphi x$  and  $x'(0) > 0$ . Then  $x$  satisfies (40) and for all  $t \geq 0$ ,*

$$X(t) \in \Delta.$$

Moreover

$$\lim_{t \rightarrow \infty} x'(t) = \|\varphi\|_\infty.$$

*Proof:* Define the trajectory  $\Phi = \{X(t) \in \mathbb{R}_+^2 : t \geq 0\}$ . Since for all  $t \geq 0$ ,  $X(t)_1' = F(X(t))_1 > 0$ ,  $\Phi$  is the graph of a differentiable function  $f : [0, S) \rightarrow \mathbb{R}_+$  with  $f(0) = x'(0) > 0$ :

$$\Phi = \{(s, f(s)) : s \in [0, S)\},$$

with  $S = \lim_{t \rightarrow \infty} x(t) \in (0, \infty]$ . Moreover

$$f'(s) = \frac{F((s, f(s)))_2}{F((s, f(s)))_1} = \frac{f(s) - \varphi(s)}{f(s)}. \quad (44)$$

The graph of the function  $\varphi$  is the curve  $L = \{(s, \varphi(s)) : s \in \mathbb{R}_+\}$  and the set

$$\Delta' = \{(x_1, x_2) \in [0, \infty)^2 : x_2 < \varphi(x_1)\}$$

is the set of points below  $L$ . Assume that the first statement of the lemma does not hold. Then by (42) and the intermediate value Theorem, the curves  $L$  and  $\Phi$  intersect. Then the exists  $s_0 > 0$  such that

$$f(s_0) = \varphi(s_0).$$

From (44),  $f'(s_0) = 0$  while  $\varphi'(s_0) > 0$ . It follows that  $(s, f(s)) \in \Delta'$  for all  $s \in (s_0, s_1)$  for some  $s_1 > s_0$ . By induction, since  $f'(s) < 0$  on  $\Delta'$  while  $\varphi'(s) > 0$ , we get that all  $s > s_0$ ,  $(s, f(s)) \in \Delta'$ .

On the other hand, since  $f'(s) < 0$ , for all  $s > s_1$ ,  $f(s) < f(s_1) < \varphi(s_1)$ . If  $x(t_1) = s_1$ , this in turn yields to : for all  $t > t_1$ ,  $x''(t) = x'(t) - \varphi(x(t)) < f(s_1) - \varphi(s_1) = -\delta$ . Integrating, this implies that  $\lim_{t \rightarrow \infty} x'(t) = -\infty$  which contradicts  $x'(t) > 0$ .

We have proved so far that for all  $t \geq 0$ ,  $X(t) \in \Delta$ . This implies that  $x'(t)$  is increasing. In particular  $\lim_{t \rightarrow \infty} x(t) = \infty$  and  $S = \infty$ . Also,  $x'$  being upper bounded by  $\|\varphi\|_\infty$ , it must converge to a limit and  $\lim_{t \rightarrow \infty} x''(t) = 0$ . Since

$$\lim_{s \rightarrow \infty} \varphi(s) = \|\varphi\|_\infty,$$

we obtain the claimed statement. ■

### 5.3 Comparison of second order differential equations

For two functions  $\psi_1, \psi_2$  in  $\mathcal{D}$ , we write  $\psi_1 \leq \psi_2$  if for all  $t \geq 0$ ,  $\psi_1(t) \leq \psi_2(t)$ .

**Lemma 20.** *Let  $\varphi \in \mathcal{D}_0$  and  $x \in \mathcal{H}$  such that  $x = T_\varphi x$  and  $x'(0) > 0$ . Let  $\psi \in \mathcal{D}$  and  $y$  be a solution of  $y'' - y' + \psi(y) = 0$  with  $y(0) = 0$ ,  $y'(0) > 0$  and. We define the exit times*

$$T = \inf\{t \geq 0 : (y(t), y'(t)) \notin \mathbb{R}_+^2\},$$

$$T_+ = \inf\{t \geq 0 : y'(t) \geq \|\varphi\|_\infty\} \quad \text{and} \quad T_- = \inf\{t \geq 0 : \varphi(y(t)) \leq y'(t)\}.$$

(i) *If  $T_+ < T$ ,  $T_+ < \infty$  and  $\varphi \leq \psi$  then  $y'(0) \geq x'(0)$ .*

(ii) *If  $T_- < T$ ,  $T_- < \infty$  and  $\psi \geq \varphi$  then  $x'(0) \geq y'(0)$ .*

*Proof:* Let us start with the hypothesis of (i). The proof is by contradiction : we also assume that  $y'(0) < x'(0)$ . We set  $Y(t) = (y(t), y'(t))$  and  $G(y_1, y_2) = (y_2, y_2 - \psi(y_1))$ . Define the trajectories  $\Phi = \{X(t) \in \mathbb{R}_+^2 : t \geq 0\}$ , and for  $\tau > 0$ ,  $\Psi(\tau) = \{Y(t) \in \mathbb{R}_+^2 : 0 \leq t \leq \tau\}$ . By Lemma 19,  $\Phi$  is the graph of an increasing function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $f(0) = x'(0) > 0$  :

$$\Phi = \{(s, f(s)) : s \in \mathbb{R}_+\}.$$

Similarly, if  $y \in \mathbb{R}_+^2$ ,  $G(y)_1 = y_2 > 0$ . Thus there exists a differentiable function  $g : [0, y(T)] \rightarrow \mathbb{R}_+$  such that

$$\Psi(T) = \{(s, g(s)) : s \in [0, y(T)]\},$$

with

$$g'(s) = 1 - \frac{\psi(s)}{g(s)}.$$

Now, the assumption  $0 < y'(0) < x'(0)$  reads  $0 < g(0) < f(0)$ .

By assumption, there is a time  $S > 0$  such that  $g(S) \geq \|\varphi\|_\infty > f(S)$ . Hence, by the intermediate value Theorem, there exists a first time  $0 < s_1 < S$  such that the curves intersect :  $g(s_1) = f(s_1)$  and  $g(s) < f(s)$  on  $[0, s_1)$ . However, it follows from (44) that for  $s \in [0, s_1)$ ,

$$g'(s) = 1 - \frac{\psi(s)}{g(s)} < 1 - \frac{\varphi(s)}{f(s)} = f'(s).$$

Hence, integrating over  $[0, s_1]$ , we find  $g(s_1) - g(0) = f(s_1) - g(0) < f(s_1) - f(0)$ . This contradicts  $g(0) < f(0)$ . We have proved (i). The proof of (ii) is identical and is omitted. ■

### 5.4 Proof of Theorem 7

We first linearize (40) with  $\varphi(s) = \lambda(1 - e^{-s})$  in the neighborhood of  $\lambda = 1/4$ .

**Step one : Linearization from above.** We have  $\varphi'(0) = \lambda$ , and from the concavity of  $\varphi$ ,

$$\varphi(s) \leq \lambda s. \quad (45)$$

We take  $\lambda > 1/4$  and consider the linearized differential equation

$$y'' - y' + \lambda y = 0. \quad (46)$$

The solutions of this differential equation are

$$y(t) = a \sin(\omega t) e^{\frac{t}{2}} + b \cos(\omega t) e^{\frac{t}{2}},$$

with

$$\omega = \sqrt{\lambda - \frac{1}{4}}.$$

We use this ODE to upper bound  $x'_\lambda(0)$ .

**Lemma 21.** *For all  $\lambda > 1/4$ ,*

$$x'_\lambda(0) \leq \frac{e^2}{4} e^{-\frac{\pi}{2\omega}} (1 + O(\omega^2)).$$

*Proof:* Let  $a > 0$  and consider the function

$$y(t) = a \sin(\omega t) e^{\frac{t}{2}}.$$

We have  $y(0) = 0$ ,  $y'(0) = a\omega$ ,

$$y'(t) = a e^{\frac{t}{2}} (\omega \cos(\omega t) + \frac{1}{2} \sin(\omega t)),$$

and

$$y''(t) = a e^{\frac{t}{2}} \left( \omega \cos(\omega t) + \left( \frac{1}{4} - \omega^2 \right) \sin(\omega t) \right).$$

Define

$$T = \frac{\pi}{\omega} - \frac{1}{\omega} \arctan \left( \frac{\omega}{\frac{1}{4} - \omega^2} \right) = \frac{\pi}{\omega} - 4 + O(\omega^2).$$

On the interval  $[0, T]$ ,  $y''(t) \geq 0$  and  $y''(T) = 0$ . Thus the function  $y'(t)$  is increasing on  $[0, T]$  and

$$y'(T) = e^{-2} a e^{\frac{\pi}{2\omega}} (\omega + O(\omega^3)).$$

Hence, we may choose  $a$  such that  $y'(T) = \lambda = \frac{1}{4} + \omega^2$  with

$$a = \frac{e^2}{4} \frac{e^{-\frac{\pi}{2\omega}}}{\omega} (1 + O(\omega^2)).$$

It remains to use (45) with Lemma 20. ■

**Step two : linearization from below.** For  $0 < \kappa < 1$ , we define

$$\ell = \frac{1}{4} + \kappa^2 \omega^2 < \lambda,$$

and the function in  $\mathcal{D}$

$$\psi(s) = \min(\ell s, \varphi(s)).$$

In particular

$$\varphi \geq \psi. \tag{47}$$

We shall thus consider the linear differential equation

$$y'' - y' + \ell y = 0, \tag{48}$$

The solutions of (48) are

$$y(t) = a \sin(\omega \kappa t) e^{\frac{t}{2}} + b \cos(\omega \kappa t) e^{\frac{t}{2}}.$$

A careful choice of  $a, \kappa$  will lead to the following lower bound.

**Lemma 22.** *For all  $\lambda > 1/4$ ,*

$$x'_\lambda(0) \geq \frac{8e}{\pi} \omega^3 e^{-\frac{\pi}{2\omega}} (1 + O(\omega^2)).$$

*Proof:* For  $a > 0$ , we look at the solution

$$y(t) = a \sin(\omega \kappa t) e^{\frac{t}{2}}.$$

We have  $y(0) = 0$ ,  $y'(0) = a\kappa\omega$ .

$$y'(t) = a e^{\frac{t}{2}} (\omega \kappa \cos(\omega \kappa t) + \frac{1}{2} \sin(\omega \kappa t)).$$

We repeat the argument of Lemma 21. On the interval  $[0, T]$ ,  $y''(t) \geq 0$  and  $y''(T) = 0$ , where

$$T = \frac{\pi}{\omega \kappa} - \frac{1}{\omega \kappa} \arctan \left( \frac{\omega \kappa}{\frac{1}{4} - \omega^2 \kappa^2} \right) = \frac{\pi}{\omega \kappa} - 4 + O(\omega^2),$$

and the  $O(\cdot)$  is uniform over all  $\kappa > 1/2$ . The function  $y'(t)$  is increasing on  $[0, T]$  and

$$y'(T) = a e^{-2} e^{\frac{\pi}{2\omega}} \omega \kappa (1 + O(\omega^2)).$$

Now, we have  $\ell s \leq \varphi(s)$  for all  $s \in [0, \sigma]$  with

$$\ell \sigma = \lambda (1 - e^{-\sigma}).$$

It yields to

$$\sigma = 2 \left( 1 - \frac{\ell}{\lambda} \right) + O \left( 1 - \frac{\ell}{\lambda} \right)^2 = 8(1 - \kappa^2) \omega^2 + O((1 - \kappa^2) \omega^4).$$

However from (48), for  $t = T$ , since  $y''(T) = 0$ , we have

$$\frac{y'(T)}{y(T)} = \ell = \frac{1}{4} + \kappa^2 \omega^2.$$

We may choose  $a$  such that  $y(T) = \sigma$  by setting

$$a = \sigma \frac{e^2}{4} \frac{e^{-\frac{\pi}{2\omega\kappa}}}{\omega\kappa} (1 + O(\omega^2)) = 2e^2 e^{-\frac{\pi}{2\omega\kappa}} \frac{(1 - \kappa^2)\omega}{\kappa} (1 + O(\omega^2)).$$

Now, in the domain  $0 \leq y \leq \sigma$  the non-linear differential equation  $y'' - y' + \psi(y)$  obviously coincides with (48). Thus, using (47) and Lemma 20, it yields to

$$x'(0) \geq y'(0) = 2e^2 e^{-\frac{\pi}{2\omega\kappa}} (1 - \kappa^2) \omega^2 (1 + O(\omega^2)).$$

Taking finally  $\kappa = 1 - 2\omega/\pi$  gives the statement. ■

**Step three : End of proof.** We now complete the proof of Theorem 7. We start with the left hand side inequality. By (42), we have

$$x(t) \geq x'(0)t.$$

It follows from (36) that

$$q(\lambda) = \int_0^\infty e^{-x(t)} e^{-t} dt \leq \int_0^\infty e^{-x'(0)t} e^{-t} dt = \frac{1}{1 + x'(0)}.$$

It remains to use Lemma 22 and we obtain the left hand side of Theorem 7.

We now turn the right hand side inequality. For  $X = (x_1, x_2) \in \mathbb{R}^2$ , define  $G(X) = (x_2, x_2)$ . From the definition of  $F$  in (43), we have, component-wise, for any  $X \in \mathbb{R}^2$ ,

$$F(X) \leq G(X).$$

Note also that  $G$  is monotone : if component-wise  $X \leq Y$  then  $G(X) \leq G(Y)$ . It follows easily that if  $X(0) = Y(0)$ ,  $X' = F(X)$  and  $Y' = G(Y)$  then

$$X(t) \leq Y(t).$$

Looking at the solution of  $y'' - y' = 0$  such that  $y(0) = 0$  and  $y'(0) = x'(0)$ , we get that

$$x(t) \leq x'(0)(e^t - 1).$$

We deduce from (36) that, for any  $T > 0$ ,

$$\begin{aligned} q(\lambda) &= \int_0^\infty e^{-x(t)} e^{-t} dt \geq \int_0^T e^{-x'(0)(e^t - 1)} e^{-t} dt \\ &\geq \int_0^T (1 - x'(0)(e^t - 1)) e^{-t} dt \\ &\geq 1 - e^{-T} - x'(0)T. \end{aligned}$$

Now, we notice that in order to prove Theorem 7, by Lemma 21, we may choose  $\lambda$  close enough to  $1/4$  so that  $x'(0) < 1$ . We finally take  $T = -\ln(x'(0))$  and apply Lemma 21. This concludes the proof of Theorem 7. ■

## Appendix

In this appendix, for the sake of completeness we include the proof of the following lemma on Galton-Watson trees.

**Lemma 23.** *Let  $T$  be a Galton-Watson tree with mean number of offsprings  $d > 1$ . Conditioned on  $T$  is infinite,  $T$  is a.s. lower  $d$ -ary.*

*Proof:* Let  $1 < \delta < d$  and  $Z_n = |V_n|$  be the number of offsprings of generation  $n$ . From Seneta-Heyde Theorem (see [20, Chapter 5]), conditioned on  $T$  is infinite, a.s.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = d.$$

Let  $p > 0$  be the probability that  $T$  is infinite. It implies that for any  $\varepsilon > 0$ , for all  $n$  large enough, we have  $\mathbb{P}(Z_n \geq \delta^n) \geq p - \varepsilon$ .

Now, consider a new Galton-Watson tree  $T'$  starting from the root  $\emptyset$  where each vertex produces independently  $m = \lfloor \delta^n \rfloor$  offsprings with probability  $p - \varepsilon$  and 0 offspring otherwise. From what precedes, we may couple  $T$  and  $T'$  such that  $T'$  is a subtree of  $T^{*n}$ .

We are now going to prove that  $T'$  contains a large regular tree with probability at least  $p - 2\varepsilon$ . To this end, we set

$$q = q(\varepsilon) = 1 - p + \varepsilon.$$

Note that we may have chosen  $n = n(\varepsilon)$  large enough so that

$$q + (1 - q)e^{-m\varepsilon^2/2} \leq q + \varepsilon. \quad (49)$$

We consider the following pruning algorithm on  $T'$ . At step 0, we start with all vertices of  $T'$ . At step 1, we remove all vertices which have less than  $(1 - q - 2\varepsilon)m$  offsprings. We now iterate: at step  $k \geq 1$ , we remove all vertices which have less than  $(1 - q - 2\varepsilon)m$  offsprings left by step  $k - 1$ .

Denote by  $\rho_k$  the probability that the root of  $T'$  is removed by step  $k$ . We have  $\rho_0 = 0$ ,  $\rho_1 = q$  and  $(\rho_k)_{k \geq 0}$  is a non-decreasing sequence. We are going to check by recursion that for all  $k \geq 1$ ,

$$\rho_k \leq q + \varepsilon. \quad (50)$$

Indeed, let  $k \geq 1$  and assume that  $\rho_{k-1} \leq q + \varepsilon$ . Note that if the root is removed by step  $k$ , then either it has 0 offspring or more than  $(q + 2\varepsilon)m$  of its offsprings were removed by step  $k - 1$ . From the recursive structure of the Galton-Watson tree, the probability that an offspring was removed by step  $k - 1$  is  $\rho_{k-1}$  and these events for each offspring are independent. It follows that

$$\rho_k \leq q + (1 - q)\mathbb{P}\left(\sum_{i=1}^m X_i \geq (q + 2\varepsilon)m\right),$$

where  $(X_i)_{1 \leq i \leq m}$  are i.i.d.  $\{0, 1\}$ -Bernoulli variables with mean  $\rho_{k-1}$ . By recursion hypothesis,  $\rho_{k-1} \leq q + \varepsilon$ . Hence, Hoeffding's inequality leads to

$$\mathbb{P}\left(\sum_{i=1}^m X_i \geq (q + 2\varepsilon)m\right) \leq \mathbb{P}\left(\sum_{i=1}^m (X_i - \mathbb{E}X_i) \geq \varepsilon m\right) \leq e^{-m\varepsilon^2/2}.$$



From (49), we deduce that  $\rho_k \leq q + \varepsilon$ . This proves (50).

We have thus proven that with probability at least  $1 - (q + \varepsilon) = p - 2\varepsilon$ , the root of  $T$  is never removed by the pruning algorithm. However, on the latter event, by construction  $T'$  contains a  $\lfloor (1 - q - 2\varepsilon)m \rfloor$ -ary tree rooted at  $\emptyset$  (note that  $1 - q - 2\varepsilon = p - 3\varepsilon$ ).

We may now conclude the proof. We apply the above argument to some  $\delta' \in (\delta, d)$ . This proves that for any  $0 < \varepsilon < 1$ , there exists an integer  $n_\varepsilon$  such that with probability at least  $(1 - \varepsilon)p$ ,  $T^{*n_\varepsilon}$  contains a  $\lceil \delta^{n_\varepsilon} \rceil$ -ary tree. Note that the latter event is contained in the event that  $T$  is infinite. It follows that the conditional probability that  $T^{*n_\varepsilon}$  contains a  $\lceil \delta^{n_\varepsilon} \rceil$ -ary tree, given  $T$  infinite, is at least  $1 - \varepsilon$ .

We finally consider the sequence  $\varepsilon_k = 1/k^2$ . From Borel-Cantelli lemma, conditioned on  $T$  infinite, a.s. there exists  $k$  such that  $T^{*n_{\varepsilon_k}}$  contains a  $\lceil \delta^{n_{\varepsilon_k}} \rceil$ -ary tree. ■

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